

Stability, as Told by its Developers*

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* A. Loría and E. Panteley, chapter in *Advanced topics in control systems theory*, Lecture Notes in Control and Information Sciences, A. Loría, F. Lamnabhi-Lagarrigue, E. Panteley, eds., London: Springer Verlag, 2006.

Stability, Told by it Developers

“The authors of the present manuscript would like to insist on the fact that only the attentive reading of the original documents can contribute to correct certain errors endlessly repeated by different authors.”

J. J. Samueli & J. C. Boudenot^a

^a Translated from *H. Poincaré (1854-1912), physicien*, Editions Ellipses: Paris, 2005. The citation is taken from the epilogue of the mentioned biography of *the last universalist* –as his biographers call H. Poincaré. The authors give interesting evidence of H. Poincaré’s shared discovery – with Lorentz – of restrained relativity – cf. Comptes Rendus de l’Académie des Sciences, Paris 9th/June/1905.

Stability, generally speaking

Taken from Rouche/Mawhin [55] –see also [56].

[Consider] a solution of a differential equation representing a physical phenomenon or the evolution of some system [...] There always exist two sources of uncertainty in the initial conditions. Indeed, when one attempts to repeat a given experiment, the reproduction of the initial conditions is never entirely faithful: for instance, a satellite can only be placed in orbit from one point and with a velocity that depends on the variable circumstances related to the launching of the rockets [...] It is thus fundamental to be able to recognise the circumstances under which small variations in the initial conditions will only introduce small variations in what follows of the phenomenon.



Stability, generally speaking

- Abuse of notation: “a system is stable...”
- Stability is a property of the solutions of differential equations by which, given a “reference” solution $x^*(t, t_o^*, x_o^*)$ of

$$\dot{x} = f(t, x), \quad x_o^* = x(t_o^*, t_o^*, x_o^*) \in \mathbb{R}^n, \quad t \geq t_o^*, \quad t_o^* \geq 0,$$

any other solution $x(t, t_o, x_o)$ starting close to $x^*(t, t_o^*, x_o^*)$ (i.e. such that $t_o^* \approx t_o$ and $x_o^* \approx x_o$), remains close to $x^*(t, t_o^*, x_o^*)$ for later times.

- Theorem on continuity of solutions with respect to initial conditions establishes sufficient conditions for a **perturbed** solution to remain “close” to an unperturbed solution over a **finite** interval of time.
- Question of stability: “*small variations in the initial conditions [will] only introduce small variations in what follows of the phenomenon*”

Stability, generally speaking

- Solutions of differential equations are commonly referred to as “trajectories”;

Following [14, Hahn '59, p. 1], we say that

“a point of the real, n -dimensional space shall be denoted by the coordinates x_1, \dots, x_n . [...] In addition to the n -dimensional x -space which is also called *phase space*, we shall refer to the $(n + 1)$ -dimensional space of the quantities x_1, \dots, x_n, t , which will be called *motion space*. [...]

The notation $x = x(t)$ indicates that the components x_i of x are functions of t . If these functions are continuous, then the point $(x(t), t)$ of the motion space moves along a segment of a curve as t runs from t_1 to t_2 , [...]

The projection of a motion upon the phase space is called the *phase curve*, or *trajectory*, of the motion. In this case the quantity t plays the role of a curve parameter. ●

Types of stability

- Lagrange Stability
- Dirichlet's Stability
- Lyapunov Stability
- Input-Output Stability
- Hyperstability
- Input-to-State Stability
- Differential inclusions
- Difference equations
- Partial differential equations
- *etc.*

Lagrange and Lagrange's stability

“Messieurs DE LA PLACE, COUSIN, LE GENDRE et moi, ayant rendu compte d'un Ouvrage intitulé: Méchanique analitique, par M. DE LA GRANGE, l'Académie a jugé cet Ouvrage digne de son approvation, et d'être imprimé sous son Privilège.

Je certifie cet Extrait conforme aux registres de l'Académie. A Paris, ce 27 février 1788.

LE MARQUIS DE CONDORCET”

Lagrange and Lagrange's stability

(Cited and translated from [27, pp. 69–70]) In a system of bodies in equilibrium, the forces P, Q, R, \dots , stemming from gravity, are, as one knows, proportional to the masses of the bodies and, consequently, constant; and the distances p, q, r, \dots meet at the centre of Earth. One will thus have, in such case,

$$\Pi = Pp + Qq + Rr + \dots;$$

[...] If one now considers the same system in motion, and let u', u'', u''', \dots be the velocities, and m', m'', m''', \dots be the respective masses of the different bodies that constitute it [the system in motion], the so well-known principle of *conservation of living forces* [...] yield this equation:

$$m'u'^2 + m''u''^2 + m'''u'''^2 + \dots = \text{const.} - 2\Pi.$$



Lagrange and Lagrange's stability

The concept of equilibrium

(Cited and translated from [27, p. 70]) Hence, since in the state of equilibrium, the quantity Π is a minimum or a maximum, it follows that the quantity $m'u'^2 + m''u''^2 + m'''u'''^2 + \dots$, which represents the living force of the whole system, will be at same time a minimum or a maximum; this leads to the following principle of Statics, that, *from all the configurations that the system takes successively, that in which it has the largest or the smallest living force, is that where it would be necessary to place it [the system] initially so that it stayed in equilibrium. (See the Mémoires de l'Académie des Sciences de 1748 et 1749.)*^a ●

^a After J. Bertrand, editor of the 3rd edition of Lagrange's treatise, Lagrange had attributed in [9], the principle on Statics to the the “*little-known geometrician Courtivron*”; Lagrange removed Courtivron's name from the second edition to substitute it with the date of publication.

Lagrange and Lagrange's stability

On the stability of the equilibrium

(Cited and translated from Lagrange's treatise [[27](#), p. 71])

[...] we will show now that if this function $[\Pi]$ is a minimum, the equilibrium will have stability, that is to say, if the system being supposed initially at the state of equilibrium and then being, no matter how little, displaced from such state, it will tend itself to come back to that position while making infinitely small oscillations: on the contrary, in the case that the same function will be a maximum, the equilibrium will have no stability, and once perturbed, the system will be able to make oscillations that will not be very small, and that may make it to drift farther and farther from its initial state. ●

Lagrange's stability

Definition 1 (Lagrange's original stability) Consider a mechanical system with state $[q, \dot{q}]$. We say that the point $q = 0$ is stable if for any (infinitely small) $\delta > 0$ and $t_o \geq 0$

$$|q(t_o)| \leq \delta \implies |q(t)| \rightarrow 0 \quad \forall t \geq t_o.$$

●

- Lagrange's stability states that "[the system] will tend itself to come back to that [equilibrium] position"; **(attractivity)**

Definition 2 (Lagrange's "interpreted" stability) Consider a mechanical system with state $[q, \dot{q}]$. We say that the point $q = 0$ is stable if for any (infinitely small) $\delta > 0$ and $t_o \geq 0$ there exists $\varepsilon > 0$ such that

$$|q(t_o)| \leq \delta \implies |q(t)| \leq \varepsilon \quad \forall t \geq t_o.$$

–Dirichlet, etc.

Dirichlet's stability

(Cited and translated from [29, p. 457])

The function of coordinates depends only on the nature of forces and can be expressed by a defined number of independent variables

$\lambda, \mu, \nu, \dots,$

$$\sum mv^2 = \varphi(\lambda, \mu, \nu, \dots) + C$$

[...] the condition that expresses that [...] the system is at an equilibrium position, coincides with that which expresses that for these same values [of the coordinates], the total derivative of φ is zero; hence, for each equilibrium position, the function will be a maximum or a minimum. If a maximum really takes place, then the equilibrium is stable, that is, if one displaces infinitely little the points [coordinates] of the system from their initial values, and we give to each a small initial velocity, in the whole course of the motion the displacements of the points of the system, with respect to their equilibrium position, will remain within certain limits [that are] defined and very small. ●

Dirichlet's stability

Dirichlet speaks of a *maximum* of the function $\varphi(\lambda, \mu, \nu, \dots)$ corresponding to a *stable* equilibrium; this makes sense if we consider that in modern notation the potential energy corresponds to $-\varphi$ and the independent coordinates λ, μ, ν, \dots correspond to the generalised coordinates of a Lagrangian system (see e.g. [13]).

Definition 3 (Dirichlet's stability) Let $x := [q^\top, \dot{q}^\top]$. We say that the point $q = 0$ is stable if for each (infinitely small) $\delta > 0$ and $t_o \geq 0$ there exists an (infinitely small) $\varepsilon > 0$ such that

$$|x(t_o)| \leq \delta \implies |q(t)| \leq \varepsilon \quad \forall t \geq t_o.$$

Definition 4 (Lagrange's "interpreted" stability) Consider a mechanical system with state $[q, \dot{q}]$. We say that the point $q = 0$ is stable if for any (infinitely small) $\delta > 0$ and $t_o \geq 0$ there exists $\varepsilon > 0$ such that

$$|q(t_o)| \leq \delta \implies |q(t)| \leq \varepsilon \quad \forall t \geq t_o.$$

Dirichlet's theorem on stability

(Cited and translated from [29, p. 459])

[...] the equilibrium position corresponds to the values $\lambda = 0$, $\mu = 0$, ..., we will also suppose that $\varphi(0, 0, 0, \dots) = 0$; [...] hence,

$$\sum mv^2 = \varphi(\lambda, \mu, \nu, \dots) - \varphi(\lambda_o, \mu_o, \nu_o, \dots) + \sum mv_o^2.$$

[...] then we can easily show that, if we take $\lambda_o, \mu_o, \nu_o, \dots$ numerically smaller than l, m, n, \dots , and at same time one satisfies the inequality

$$-\varphi(\lambda_o, \mu_o, \nu_o, \dots) + \sum mv_o^2 < p,$$

each of the variables λ, μ, ν, \dots will remain during the complete duration of the motion below the limits l, m, n, \dots . •

Dirichlet's theorem on stability

Dirichlet's proof can be explained in modern terms using the total energy function, in terms of generalised positions $q := \lambda, \mu, \nu, \dots$ and velocities \dot{q} , *i.e.*

$$V(q, \dot{q}) := T(q, \dot{q}) + U(q)$$

where $T(q, \dot{q}) := \sum m v^2$ and $U(q) := -\varphi(\lambda, \mu, \nu, \dots)$, *i.e.* in general v depends on the generalised velocities and positions and the potential energy is assumed to depend only on the positions. As Dirichlet points out, we can assume without loss of generality that $U(0) = 0$. Dirichlet then poses

$$p := \min\{U(q) : |\lambda| = l, |\mu| = m, |\nu| = n \dots\}.$$

Dirichlet's theorem on stability

Now, consider initial positions $q(t_0)$ and velocities $\dot{q}(t_0)$ such that $V(q(t_0), \dot{q}(t_0)) < p$, the equation of living forces (principle of energy conservation) is

$$V(q(t), \dot{q}(t)) = V(q(t_0), \dot{q}(t_0)) \quad \forall t \geq t_0$$

so we have, necessarily, $V(q(t), \dot{q}(t)) < p$ for all $t \geq t_0$. Equivalently, $T(q(t), \dot{q}(t)) + U(q(t)) < p$ for all $t \geq t_0$. If any of the values λ, μ, ν, \dots came to overpass its respective limit, say at $t = t^*$, we would have $U(q(t^*)) \geq p$ and, necessarily, $T(q(t^*), \dot{q}(t^*)) < 0$ which is impossible.

Lagrange stability (in modern terms)

- The terminology “*stability in the sense of Lagrange*” is attributed, by Hahn [14, p. 129], to La Salle [16]; in the latter one reads:

“the boundedness of all solutions for $t \geq 0$ is also a kind of stability, called Lagrange stability”.

- Consider the system

$$\dot{x} = F(t, x) \tag{1}$$

where F is continuous, and $F(t, \cdot)$ is locally Lipschitz, uniformly in t and $F(t, 0) \equiv 0$.

Definition 5 (Lagrange stability) *The system (1) is said to be Lagrange stable if for each $\delta > 0$ and $t_o \geq 0$ there exists $\varepsilon > 0$ such that*

$$|x(t_o)| \leq \delta \implies |x(t)| \leq \varepsilon \quad \forall t \geq t_o \geq 0.$$



A theorem on Lagrange stability

- Theorems on boundedness of solutions can be found, e.g., in the texts of Yoshizawa [66], Rosier [3] and Leonov [30]. The following result is from [16].

Theorem 4—A Lagrange Stability Theorem

Let Ω be a bounded neighbourhood of the origin and let Ω^c be its complement (Ω^c is the set of all points outside Ω). Assume that $W(x)$ is a scalar function with continuous first partials in Ω^c and satisfying:

- 1) $W(x) > 0$ for all x in Ω^c ,
- 2) $\dot{W}(x) \leq 0$ for all x in Ω^c ,
- 3) $W(x) \rightarrow \infty$ as $\|x\| \rightarrow \infty$.

Then each solution of (2) $\left[\begin{array}{l} \dot{x} = X(x) \end{array} \right]$ is bounded for all $t \geq 0$.

- Another interpretation is provided by Rouché –[57]: Dirichlet's stability is seen as stability with respect to part of coordinates.

Lyapunov's stability

*"J'ai seulement eu en vue d'exposer dans cet Ouvrage ce que je suis parvenu à faire en ce moment et ce qui, peut-être, pourra servir de point de départ pour d'autres recherches de même genre."**

A. M. Liapounoff, 1907

*I only had the purpose of exposing in this Work what I managed to do at this time and what, perhaps , will be the starting point for other similar research.

Lyapunov's treatise (main) editions

- [A] A. M. Lyapunov (1892), *Obschaya zadacha ob ustoichivosti dvizheniya. Coobschenya Kharkovskogo Matematicheskogo Obasestva*.
Journal title sometimes cited as: Comm. Math. Soc. Kharkov.
- [B] A. M. Liapounoff (1907), *Problème de la stabilité de mouvement. Annales de la faculté de sciences de Toulouse*, 9:203–474.
French translation of [A] made by E. Davaux and revised by the author.
- [C] A. M. Liapounoff (1947), Facsimile reprint of [B]. Published in *Ann. Math. Studies*, vol. AM-17, Princeton University Press.
- [D] A. M. Lyapunov (1950), *Obschaya zadacha ob ustoichivosti dvizheniya*. Gostehiszdats, Moscow-Leningrad. Reprint of [A]
- [E] A. M. Lyapunov (1992), The general problem of stability of motion. *Int. J. of Contr.*, 55. No. 3. English translation of [C].
- [F] A. M. Lyapunov (1996), *The general problem of stability of motion*. Academic Press, NY. Reprint of [E], edited and commented by T. Fuller.

Lyapunov's stability

Borrowing inspiration from Lagrange and Dirichlet

(Cited and translated from [34, p. 209]) Let us consider a material [physical] system with k degrees of freedom. Let

$$q_1, q_2, \dots, q_k$$

be k independent variables by which we agree to define its position.
[...]

Considering such variables as functions of time t , we will denote their first derivatives, with respect to t , by

$$q'_1, q'_2, \dots, q'_k.$$

In each problem of dynamics, [...] these functions satisfy k second-order differential equations. ●

Lyapunov's stability

Let us assume that a particular solution is found to be

$$q_1 = f_1(t), \quad q_2 = f_2(t), \dots, q_k = f_k(t),$$

in which the quantities q_j are expressed by real functions of t , [...]
To this particular solution corresponds a determined motion of our system. By comparing it [the motion] [...] to other motions of the system that are plausible under the same forces, we will call it *unperturbed motion*, and all the rest, with respect to which it is compared, will be referred to as *perturbed motions*. ●

Lyapunov's stability

Denoting by t_o an arbitrary time instant, let us denote the corresponding values of the quantities q_j , q'_j , in an arbitrary motion, by q_{j0} , q'_{j0} .

(Cited and translated from [34, p. 210])

Let

$$q_{10} = f_1(t_o) + \varepsilon_1, \quad q_{20} = f_2(t_o) + \varepsilon_2, \quad \dots, \quad q_{k0} = f_k(t_o) + \varepsilon_k,$$

$$q'_{10} = f'_1(t_o) + \varepsilon'_1, \quad q'_{20} = f'_2(t_o) + \varepsilon'_2, \quad \dots, \quad q'_{k0} = f'_k(t_o) + \varepsilon'_k,$$

where ε_j , ε'_j are real constants. [...] that we will call *perturbations*, will define a perturbed motion. ●

Lyapunov's stability

[...] let Q_1, Q_2, \dots, Q_n be given continuous and real functions of the quantities

$$q_1, q_2, \dots, q_k, \quad q'_1, q'_2, \dots, q'_k.$$

For the unperturbed motion they will become known functions of t that we will denote respectively F_1, F_2, \dots, F_n . For a perturbed motion they will become functions of the quantities

$$t, \varepsilon_1, \varepsilon_2, \dots, \varepsilon_k, \quad \varepsilon'_1, \varepsilon'_2, \dots, \varepsilon'_k.$$

When the $\varepsilon_j, \varepsilon'_j$ are equal to zero, the quantities

$$Q_1 - F_1, \quad Q_2 - F_2, \quad \dots, \quad Q_n - F_n$$

will be zero for each value of t .



Lyapunov's stability

- Lyapunov introduces *his* stability –cf. [34, p. 210]):

“if without making the constants $\varepsilon_j, \varepsilon'_j$ zero, we make them infinitely small, the question that arises is whether it is possible to assign to the quantities $Q_s - F_s$ infinitely small limits, such that these quantities never reach them in absolute value.

The solution to this question [...] depends on the nature of the considered unperturbed motion as well as on the choice of the functions Q_1, Q_2, \dots, Q_n and on the time instant t_o . Hence, [...] the answer to this question will characterise [...] the unperturbed motion, and it is such that it will express the property that we will call *stability* [...]”

Lyapunov's definition

(Cited and translated from [34, pp. 210-211])

Let L_1, L_2, \dots, L_n be positive given numbers. If for all values of these numbers, no matter how small they are, one can choose positive numbers

$$E_1, E_2, \dots, E_k \quad E'_1, E'_2, \dots, E'_k,$$

such that, the inequalities

$$|\varepsilon_j| < E_j, \quad |\varepsilon'_j| < E'_j \quad (j = 1, 2, \dots, k)$$

being satisfied, we have

$$|Q_1 - F_1| < L_1, \quad |Q_2 - F_2| < L_2, \quad \dots, \quad |Q_n - F_n| < L_n,$$

for all values of t greater than t_0 , the unperturbed motion will be called stable WITH RESPECT TO THE QUANTITIES Q_1, Q_2, \dots, Q_n ; in the opposite case, it will be called unstable with respect to the same quantities. ●

Lyapunov's definition

Lyapunov's stability beyond Lyapunov stability

- Stability in the sense of Lyapunov is defined with respect to functions of the perturbed and unperturbed motions. Let $Q : \mathbb{R}^{2k} \rightarrow \mathbb{R}^n$ (where n is not necessarily equal to $2k$) be continuous functions of the coordinates q, q' and define the functions $F : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}^n$ as

$$F(t) := Q(f(t), f'(t)) \quad \forall t \in \mathbb{R}.$$

Definition 6 (Lyapunov's original statement) *We shall say that the unperturbed motion $(t, f(t), f'(t))$ is Lyapunov stable with respect to Q , if for any (infinitely small) $\epsilon > 0$ and $t_o \in \mathbb{R}$ there exists $\delta > 0$ such that*

$$|q(t_o) - f(t_o), q'(t_o) - f'(t_o)| \leq \delta \implies |Q(q(t), q'(t)) - F(t)| \leq \epsilon.$$



On Lyapunov's definition

- when the function $Q(q, q') = q$ (i.e. $n = k$) and the unperturbed motion is the origin of the phase space, i.e. $(t, f(t), f'(t)) = (t, 0, 0)$ then, observing that $Q(0, 0) = 0$, we have $F \equiv 0$ and therefore, we recover the property of stability of part of coordinates, i.e. *for each $\epsilon > 0$ and $t_o \in \mathbb{R}$ there exists $\delta > 0$ such that*

$$|q_o, q'_o| < \delta \implies |q(t)| < \epsilon \quad \forall t \geq t_o,$$

which is called (for $t_o \geq 0$) in [57] “stability in the sense of Lagrange-Dirichlet”

- when Q corresponds to the “identity” operator, i.e. $Q(r, s) = (r, s)$ and the unperturbed motion is the origin of the phase space then we have $F \equiv 0$. In this case, Lyapunov's stability reduces to the following: *“for any $\epsilon > 0$ and $t_o \in \mathbb{R}$ there exists $\delta > 0$ such that the inequalities*

$$|q_o, q'_o| < \delta \implies |q(t, t_o, q_o, q'_o), q'(t, t_o, q_o, q'_o)| < \epsilon.$$

On Lyapunov's contribution

- Lyapunov raises the question of stability beyond the realm of physical systems, by considering the stability of *motion* for general differential equations:

(Cited and translated from [34, pp. 212])

The solution to our question depends on the study of differential equations of the perturbed motion or, in other words, of the study of the differential equations satisfied by the functions

$$Q_1 - F_1 = x_1, \quad Q_2 - F_2 = x_2, \quad \dots, \quad Q_n - F_n = x_n.$$

[...] We will assume that the number n and the functions Q_s [are] such that the order of this system is n and that can be put in the form

$$(1) \quad \frac{dx_1}{dt} = X_1, \quad \frac{dx_2}{dt} = X_2, \quad \dots, \quad \frac{dx_n}{dt} = X_n.$$



On Lyapunov stability

- From the above formulations we recover the definition of Lyapunov stability that we are used to seeing in textbooks on nonlinear systems, such as [19, p. 98], [18, p. 98], [20, p. 112], [65, p. 136] and on ordinary differential equations, e.g. [56, p. 6]:

Definition 7 (Lyapunov stability) *The origin is a stable equilibrium of Equation (1) if, for each pair of numbers $\varepsilon > 0$ and $t_o \geq 0$, there exists $\delta = \delta(t_o, \varepsilon) > 0$ such that*

$$|x(t_o)| < \delta \implies |x(t)| < \varepsilon \quad \forall t \geq t_o \geq 0. \quad (2)$$

•

On Lyapunov stability

In some texts and articles, starting at least with [15], one also finds the following definition of stability:

Definition 8 (Lyapunov stability) *The origin is a stable equilibrium of Equation (1) if for each $t_o \geq 0$ there exists $\varphi \in \mathcal{K}$ such that*

$$|x(t, t_o, x_o)| \leq \varphi(|x_o|) \quad \forall t \geq t_o \geq 0. \quad (3)$$

•

We recall, from [15], that $\varphi \in \mathcal{K}$ if it is “*defined, continuous, and strictly increasing on $0 \leq r \leq r_1$, resp. $0 \leq r < \infty$, and if it vanishes at $r = 0$: $\varphi(0) = 0$* ”. It is established in [15, p. 169] that the two definitions, 7 and 8, are equivalent. See also^a [19, p. 136], [20, p. 150].

^aStrictly speaking, in [19] and [20] the author considers the case when φ , hence δ , are independent of t_o .

Lyapunov vs. Lagrange stability

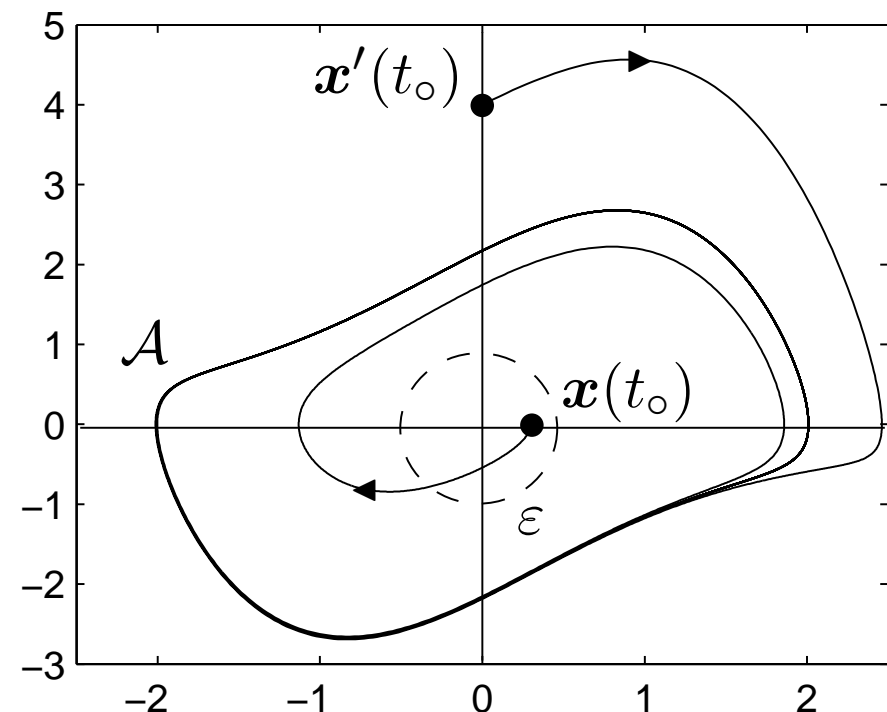
Which implies which?

Lyapunov stable systems may be Lagrange unstable and viceversa.

Example 1 Consider the van der Pol oscillator:

$$\dot{x}_1 = x_2,$$

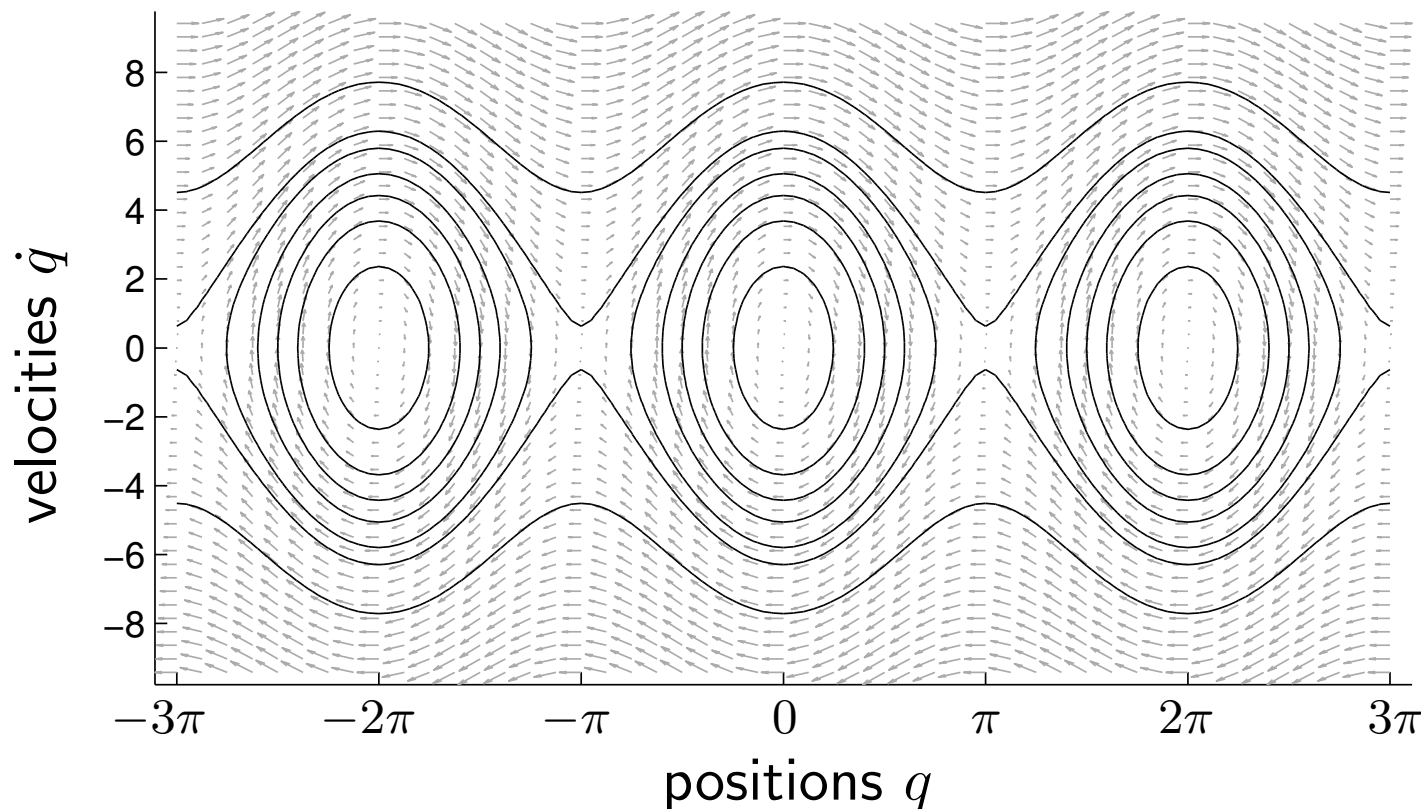
$$\dot{x}_2 = -x_1 + (1 - x_1^2)x_2$$



Lyapunov vs. Lagrange stability

Example 2 For the pendulum: $I\ddot{q} + mg\ell \sin(q) = 0$

- the origin is Lyapunov stable;
- for large (in absolute value) initial velocities, the trajectories $q(t)$ grow unboundedly; it is not Lagrange stable according to Definition 5;
- the equilibria $q = 2n\pi$, $\dot{q} = 0$ with $n \in \mathbb{Z}$ are Dirichlet stable.



Lyapunov's second method

(Cited and translated from [34, p. 222]) All the processes that we can mention to solve the question that occupies us may be classified in two categories. In the first, we shall fit all the processes that reduce to studying directly the perturbed motion and which, consequently, depend on the search for general or particular solutions of the differential equations under consideration. [...] In the second we shall fit all sort of processes that are independent of the search for solutions of the differential equations of the perturbed motion.

Such is the case, for instance, of the known process of analysis of stability of the equilibrium, in the case that there exists a function of forces.

These processes will reduce to the search of [...] functions of the variables x_1, x_2, \dots, x_n, t , whose total derivatives with respect to t , under the hypothesis that x_1, x_2, \dots, x_n are functions of t satisfying the equations (1), must satisfy such and such given conditions.

The group of processes of this category will be called *the second method*.

Lyapunov's second method

Lyapunov Functions

(Cited and translated from [34, p. 256])

we will consider real functions of real variables

$$(39) \quad x_1, x_2, \dots, x_n, t,$$

subject to the constraint

$$(40) \quad t \geq T, \quad |x_s| \leq H \quad (s = 1, 2, \dots, n).$$

We will speak of functions that, on such domain, are continuous and uniform and that are zero if

$$x_1 = x_2 = \dots x_n = 0.$$



Lyapunov's second method

Lyapunov Functions

(Cited and translated from [34, p. 257]) Let us suppose that the considered function V is such that, under the conditions (40), T being sufficiently large and H sufficiently small, it can only take values of a single sign.

Then, we shall say that it is a *function of fixed sign*; and when it will be needed to indicate its sign, we shall say that it is a *positive function* or a *negative function*.

If, moreover, the function V does not depend on t and if the constant H can be chosen sufficiently small so that, under the conditions (40), the equality $V = 0$ cannot occur unless we have

$$x_1 = x_2 = \dots x_n = 0,$$

we shall call the function V , as if it were a quadratic function, *definite function* or, trying to attract attention on its sign, *positive definite* or *negative definite*.

Lyapunov's second method

Lyapunov Functions

Concerning functions that depend on t , we shall still use these terms but then, we shall only speak of a *definite* function V under the condition that we can find a function W independent of t , that is positive definite and that in addition one of the expressions

$$V - W \quad \text{or} \quad -V - W$$

be a positive function. ●

Lyapunov's theorem on stability

(Cited and translated from [34, pp. 258-259]) Everybody knows Lagrange's theorem on the stability of the equilibrium in the case when there exists a function of the forces, as well as the elegant demonstration given by Lejeune-Dirichlet. The latter relies on considerations that may serve the demonstration of many other analogous theorems.

Guided by these considerations we will establish here the following propositions:

THEOREM I. – If the differential equations of the perturbed motion are such that it is possible to find a definite function V , whose derivative V' is a function of fixed sign and opposite to that of V , or it is exactly zero, the unperturbed motion is stable.

First sentence of the proof:

Let us suppose, to fix the ideas, that the function found V is positive definite and that its derivative V' is negative or identically zero. •

Classical theorem on Lyapunov stability

(Cited from [15, p. 102]) Consider the differential equation^a

$$(25.1) \quad \dot{\boldsymbol{x}} = f(\boldsymbol{x}), \quad 0 \leq |\boldsymbol{x}| \leq h, \quad f \in E$$

Theorem 25.1. If there exists a positive definite function $v(\boldsymbol{x})$ whose derivative $\dot{v}(\boldsymbol{x})$ for (25.1) is negative semi-definite or identically zero then the equilibrium of (25.1) is stable. •

- W. Hahn (1967), *Stability of motion*. Springer-Verlag, New York.

Asymptotic stability

Lyapunov introduced the property of asymptotic stability in a remark following the proof of his theorem on stability, –cf. [34, THEOREM I], in the following terms:

(Cited and translated from [34, p. 261])

Remark II. – If the function V , while satisfying the conditions of the theorem [THEOREM I], allows an infinitely small upper bound, and if its derivative represents a definite function, one can show that every perturbed motion, sufficiently close to the unperturbed motion, will approach the latter asymptotically. ●

- The terminology “admits an infinitely small upper bound” was common in Soviet literature at least until the 1950s; this is today referred to in the literature as “decreascent”.
- Lyapunov only says that the derivative of V should be *definite*; yet, according to [34, THEOREM I] and the way Lyapunov introduced *his* functions, it is understood that he means definite and of opposite sign to that of V' .

Asymptotic stability

- N. N. Krasovskii in [22, p. 2] says, just before presenting the definitions of stability and asymptotic stability, that “*some of the definitions of refined types of stability follow Četaev’s annotations in*” [8, pp. 11-36].
- In Lyapunov’s definition one implicitly reads that the property is *local*, that is, it is a property of the origin with respect to motions that originate in an *infinitely small* open neighbourhood of the trivial solution.
- Furasov’s definition of (local) asymptotic stability is the following:

(Cited and translated from [12, p. 13])

Definition 2.2. The unperturbed motion Σ is called *asymptotically stable*, if it is stable *a la* Lyapunov and there exists a positive constant $\Delta \leq \delta(\varepsilon, t_0)$, such that the condition

$$x(x_0, t) \rightarrow 0 \quad \text{as } t \rightarrow \infty$$

holds for all the solutions of the system, starting in the region

$$\|x_0\| < \Delta.$$



Asymptotic stability

The property that $|x_o| \leq \delta$ implies that

$$\lim_{t \rightarrow \infty} x(t, t_o, x_o) = 0$$

was sometimes called *quasi-asymptotic stability* (cf. [2, p. 142], cf., [14, p. 7]^a) or *quasi-equi-asymptotic stability* –cf. [66, p. 44] and it may be expressed by the more precise statement: $|x_o| < \delta$ implies that for each $\eta > 0$ there exists $T(\eta) > 0$ such that

$$|x(t, t_o, x_o)| < \eta \quad \forall t > t_o + T. \quad (4)$$

In general the number T depends on x_o and on t_o ; not only on η .

^aHahn uses $p(t, t_o, x_o)$ to denote the solutions.

Asymptotic stability

This brings us to the following well-adopted definition of asymptotic stability: *the equilibrium is asymptotically stable if it is stable and attractive* (cf. [65, Definition 31, p. 141], [56, 55, Definition 2.11, p. 6], [20, Definition 4.1, p. 112]). More precisely, we have:

Definition 9 (Asymptotic stability) *We say that the origin of (1) is asymptotically stable if it is stable in the sense of Definition 7 and there exists $\delta > 0$ such that, for each $\eta > 0$, $t_o \geq 0$ there exists $T(\eta, t_o) > 0$, such that*

$$|x_o| < \delta \quad \implies \quad |x(t, t_o, x_o)| < \eta \quad \forall t > t_o + T. \quad (5)$$

•

Asymptotic stability

Definition 10 *The equilibrium of the differential equation*

$$\dot{x} = f(t, x) \quad t \geq t_o$$

is said to be attractive if there exists $\eta > 0$ and, for each x_o satisfying $|x_o| < \eta$, a function σ of class \mathcal{L} such that

$$|x(t, t_o, x_o)| < \sigma(t - t_o), \quad \forall t > t_o.$$

•

We recall from [15, p. 7] that “a real function $\sigma(s)$ belongs to class \mathcal{L} if it is defined, continuous and strictly decreasing on $0 \leq s_1 \leq s < \infty$ and if $\lim \sigma(s) = 0$ ($s \rightarrow \infty$)”. Similarly as in the case of Definition 8, the function σ depends in general on t_o .

Except for slight changes in the notation, this definition is taken from [15, p. 7, Def. 2.8]

Globalization of asymptotic stability

J. P. La Salle in [16], 1960:

(Cited from [16, pp. 521–522]) [...] it is never completely satisfactory to only know that the system is asymptotically stable without some idea of the size of the region of asymptotic stability [...] Ideally, we might like to have that the system return to equilibrium regardless of the size of the [initial] perturbation. ●

In early literature (1950s) the terms *asymptotic stability in the whole* and *asymptotic stability in the large* were introduced in Soviet literature to distinguish the case when asymptotic stability holds not only for *infinitely small initial perturbations* (i.e. conditions) as originally defined by A. M. Lyapunov.

Hahn in *Stability of Motion* 1967:

(cited from [15, p. 109]) [...] if the domain of attraction is all of \mathbb{R}^n we speak of *asymptotic stability in the whole*, (cf. sec 2) or also of *global asymptotic stability* [...] ●

Global, *i.e.*, in the Large or in the Whole ?

W. Hahn, in [14], explains the difference between asymptotic stability *in the large* and asymptotic stability *in the whole* and warns against mistaken translations:

(Cited and translated from [21, p. 149]) When addressing questions of stability in the large^[1] the interest [resides on] the estimate of the domain of stability (in the case when there is no stability in the whole).

(Cited from [14, p. 8]) If relation (2.10) [here, (4)] is valid for *all* points x_0 from which motions originate, we shall say that the equilibrium is *asymptotically stable in the large* (Aizerman [1], Krasovskii [21]). If relation (2.10) [here, (4)] holds for all points of the phase space, the equilibrium is said to be *asymptotically stable in the whole* (Barbashin and Krasovskii [6, 7]). La Salle [16] proposed “complete stability.” The distinction between asymptotic stability in the large and asymptotic stability in the whole has often been obliterated by inaccurate translations of the Russian terminology. However, it becomes important in cases where Eq. (2.7) [$\dot{x} = f(t, x)$] is not defined for all points of the phase space.

La Salle's complete stability

(Cited from [16, p. 524]) For many systems it may be important to assure that no matter how large the perturbation, or in a feedback control system, regardless of the size of the error, the system tends to return to its equilibrium state. This is asymptotic stability in the large. In place of this awkward expression we shall say completely stable. The system (2) $\left[\dot{x} = X(x) \right]$ will be said to be completely stable if the origin is stable and if every solution tends to the origin as t tends to infinity. ●

Asymptotic stability in the large

Section “Stability in the large, in the whole” of Furasov’s book [12] is a rare passage dealing with both concepts in certain rigour:

(Cited and translated from [12, p. 29])

Definition 6.1. Let Δ_0 be a given positive number. The unperturbed motion Σ is called *asymptotically stable in the large*, if it is stable *a la* Lyapunov and condition (2.5) [$x(x_0, t) \rightarrow 0$ as $t \rightarrow \infty$] is satisfied for any initial perturbations x_0 from the region

$$|x_0| \leq \Delta_0.$$

•

- Asymptotic stability in the large makes good sense when speaking of attractivity for a “large” domain as opposed to an infinitesimal neighbourhood of the origin, but not all of \mathbb{R}^n .

Asymptotic stability in the large

(Cited from [25, pp. 58-59, Theorems VI and VII] together)

Let $V(x)$ be a scalar function with continuous first partial derivatives.

Let Ω_l designate the region where $V(x) < l$. Assume that Ω_l is bounded and that within Ω_l :

$$V(x) > 0 \quad \text{for} \quad x \neq 0, \quad (\text{a})$$

$$\dot{V}(x) < 0 \quad \text{for all} \quad x \neq 0 \text{ in } \Omega_l, \quad (\text{b})^*$$

then the origin is asymptotically stable, and above all, every solution in Ω_l tends to the origin as $t \rightarrow \infty$ (The last conclusion goes beyond Lyapunov's asymptotic stability theorem). ●

Asymptotic stability in the large

(Cited and translated from [6])

Let us consider the system

$$\dot{x} = -\frac{2x}{(1+x^2)^2} + 2y, \quad \dot{y} = -\frac{2y}{(1+x^2)^2} - \frac{2x}{(1+x^2)^2} \quad (2)$$

For this system the following positive-definite function will serve us as a Lyapunov function:

$$v(x, y) = y^2 + \frac{x^2}{1+x^2}.$$

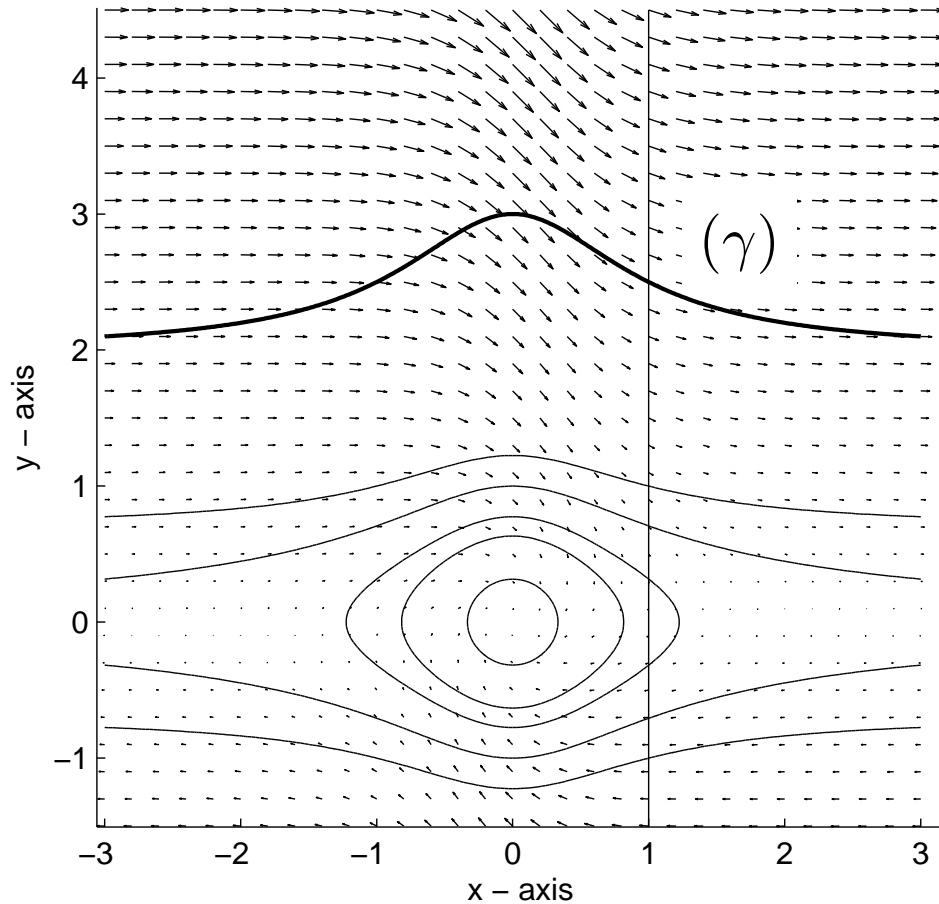
Next, we have

$$\frac{dv}{dt} = -\frac{4x^2}{(1+x^2)^4} - \frac{4y^2}{(1+x^2)^2}.$$

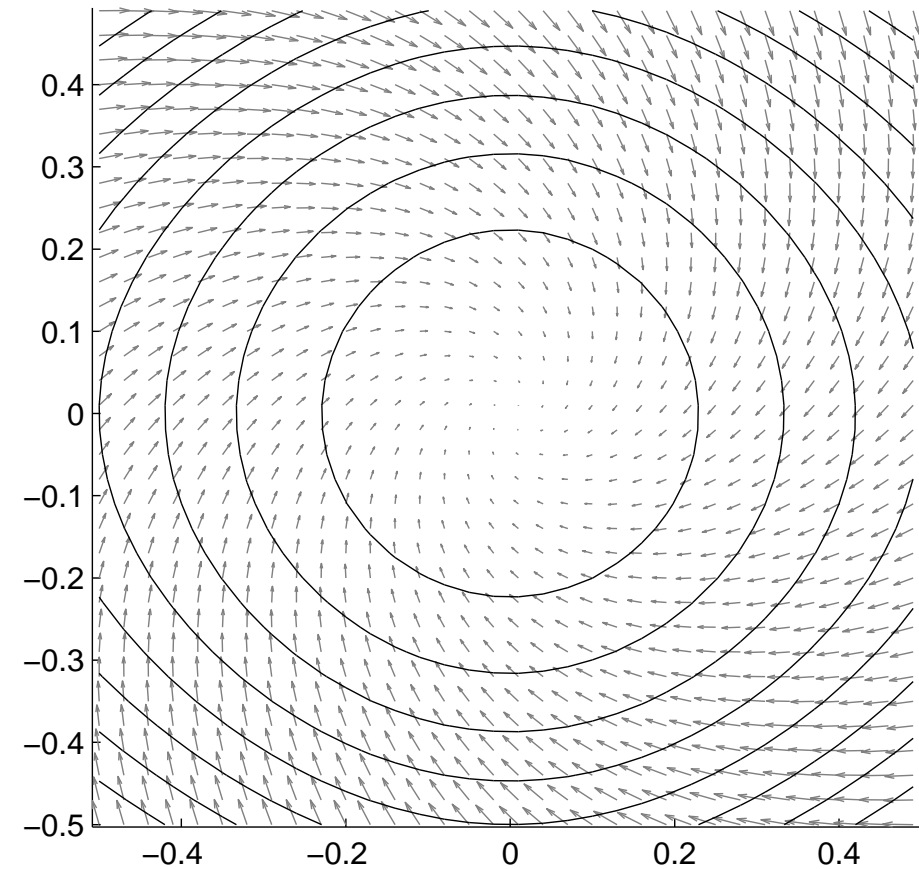
[...] we will show that on the plane (x, y) there is a set of instability for the system (2). Indeed, consider a curve (γ) given by the equation

$y = 2 + \frac{1}{1+x^2}$. Calculating $\frac{dx}{dt}$ and $\frac{dy}{dt}$ along this curve, •

Stable in the large, not in the whole



Flowchart on the phase plane.



Estimate of the region of attraction.

Asymptotic stability in the whole

From E. A. Barbashin and N. N. Krasovskii's milestone paper [6]:

(Cited and translated from [6]) We say, that the trivial solution $x_i = 0$ of systems (1)

$$\left[\frac{dx}{dt} = X_i(x_1, x_2, \dots, x_n), \quad i = 1, 2, \dots, n, \quad (1) \right]$$

is asymptotically stable for any initial perturbations if it is stable in the sense of Lyapunov (for sufficiently small perturbations) and if each other solution $x_i(t)$ of systems (1) possesses the property $\lim_{t \rightarrow \infty} x_i(t) = 0, i = 1, 2, \dots, n.$ •

(Cited from [14, p. 8]) [...] the set of all points (x_o, t_o) from which motions originate, satisfying the relation (2.10) [here, (4)], forms the *domain of attraction* of the equilibrium. •

(Cited from [15, p. 109]) If the domain of attraction is all of \mathbb{R}^n we speak of *asymptotic stability in the whole* (cf. Sec. 2) or also of *global asymptotic stability*. •

Theorems on asymptotic stability in the whole

(Cited from [20, p. 124])

Theorem 4.2 Let $x = 0$ be an equilibrium point for $\dot{x} = f(x)$. Let $V : \mathbb{R}^n \rightarrow \mathbb{R}$ be a continuously differentiable function such that

$$V(0) = 0 \text{ and } V(x) > 0, \quad \forall x \neq 0$$

$$|x| \rightarrow \infty \Rightarrow V(x) \rightarrow \infty$$

$$\dot{V}(x) < 0, \quad \forall x \neq 0$$

then $x = 0$ is globally asymptotically stable. •

The previous classical theorem, as well as its converse, were originally contributed by E. A. Barbashin and N. N. Krasovskii in [6]:

(Cited and translated from [6, p. 454])

Theorem 1. If there exists a positively definite, infinitely large function $v(x_1, x_2, \dots, x_n)$ which has definitely negative derivative then trivial solution of system (1) is asymptotically stable for any initial perturbations. •

Barbashin and Krasovskii's theorem

Original definition of GAS:

“We say, that the trivial solution $x_i = 0$ of systems (1)

$$\left[\frac{dx}{dt} = X_i(x_1, x_2, \dots, x_n), \quad i = 1, 2, \dots, n, \quad (1) \right]$$

is asymptotically stable for any initial perturbations if it is stable in the sense of Lyapunov (for sufficiently small perturbations) and if each other solution $x_i(t)$ of systems (1) possesses the property $\lim_{t \rightarrow \infty} x_i(t) = 0, i = 1, 2, \dots, n.$ ”

“Theorem 4 (1952). *Let exist an infinitely large definitely positive function $v(x_1, x_2, \dots, x_n)$ and a set M such that*

$$\frac{dv}{dt} < 0 \quad \text{not in } M; \quad \frac{dv}{dt} \leq 0 \quad \text{in } M.$$

Let the set M have the property that on any intersection of the set $v = c$ ($c \neq 0$) and M there do not exist positive semi-trajectories of system (1). We state that the trivial solution $x_i = 0$ of system (1) is asymptotically stable for any initial perturbations.”

Barbashin and Krasovskii's theorem

Illustrative example: Let us consider a system of two differential equations described by [6, Equation (1)], *i.e.*

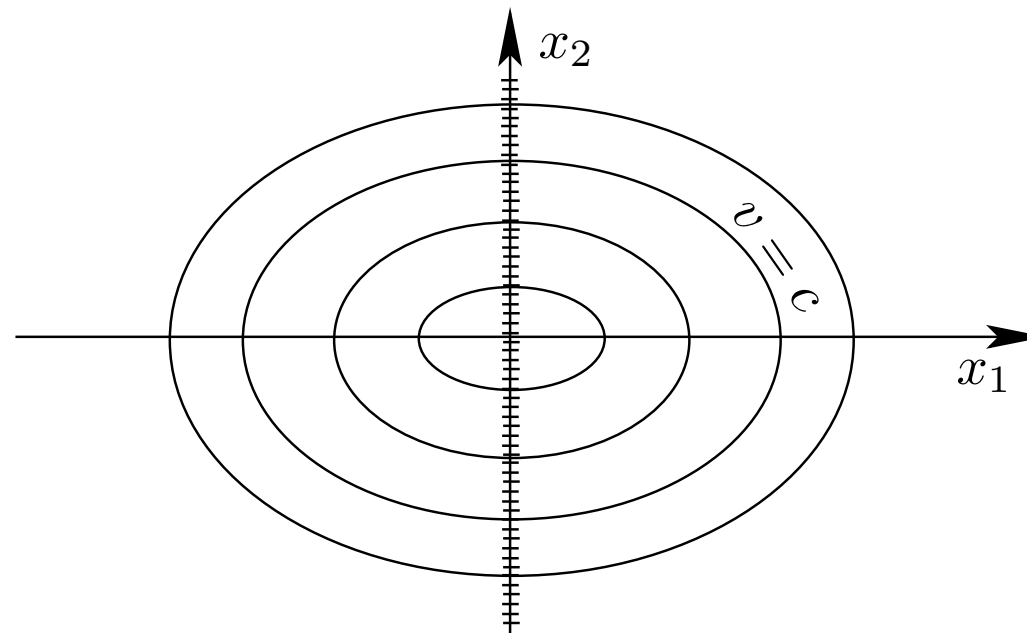
$$\frac{dx}{dt} = X_i(x_1, x_2), \quad i = 1, 2.$$

Assume that there exists:

- a positive definite, radially unbounded (infinitely large) function $v(x_1, x_2)$
- a continuous function w such that $w(0) = 0$, $w(x_1) > 0$ for all $x_1 \neq 0$ and

$$\begin{aligned} \frac{dv}{dt} &= \frac{\partial v}{\partial x_1} X_1(x_1, x_2) + \frac{\partial v}{\partial x_2} X_2(x_1, x_2) \\ &= -w(x_1). \end{aligned}$$

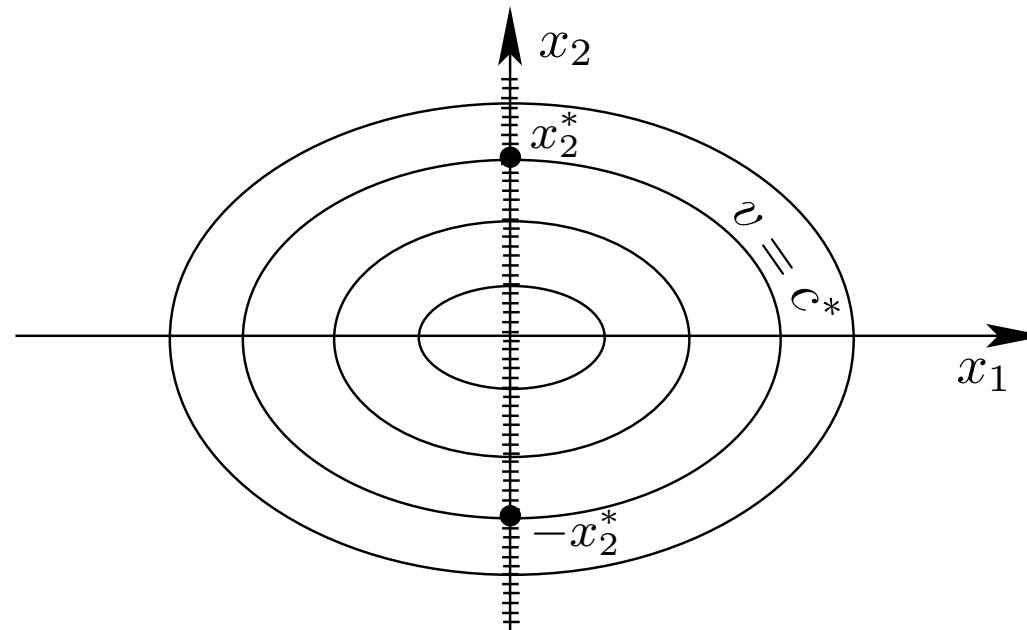
Barbashin and Krasovskii's theorem



To apply [6, Theorem 4]:

- we see that M corresponds to the vertical axis of the phase space frame;
- we must verify that the origin is the only element of the intersection $M \cap \{v = c\}$ that contains continuous positive semi-trajectories
(i.e. functions $t \mapsto x$ with $t \geq 0$)

Barbashin and Krasovskii's theorem



To apply [6, Theorem 4]:

- otherwise, we must verify that the largest invariant set $E \subset \mathbb{R}^n$ contained in $M \cap \{v = c\}$ is the origin.
- For this, fix $c = c^* > 0$ arbitrarily; in this case, $M \cap \{v = c^*\} = \{(0, -x_2^*), (0, x_2^*)\}$ with $x_2^* \neq 0$;
- does $0 = X_2(0, x_2)$ has other solutions than $\{x_2 = 0\}$?

NO \implies GAS

La Salle's “parallel” theorem

La Salle (1960) stated (without credit to Barbashin, Krasovskii):

“Theorem 3. *Let $V(x)$ be a scalar function with continuous first partials for all x . Assume that*

$$1) \ V(x) > 0 \text{ for all } x \neq 0$$

$$2) \ \dot{V}(x) \leq 0 \text{ for all } x.$$

Let E be the set of all points where $\dot{V}(x) = 0$, and let M be the largest invariant set contained in E . Then every solution of (2) $\left[\begin{array}{l} \dot{x} = X(x) \end{array} \right]$ bounded for $t \geq 0$ approaches M as $t \rightarrow \infty$.”

- Note that the invariance principle does *not* establish asymptotic stability but guarantees the attractivity of the set M , assumed invariant.
- in the particular case where $M = \{0\}$ and V is radially unbounded, we recover the well-known La Salle's theorem for global asymptotic stability and which is equivalent to Barbashin/Krasovskii's 1954 theorem.

Example: pendulum

Set-point control: Consider the pendulum equation with a PD-plus-gravity-precompensation controller, *i.e.*

$$m\ell^2\ddot{q} + mg\ell \sin(q) = -k_p\tilde{q} - k_d\dot{q} + mg\ell \sin(q_d).$$

The state-space representation is

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ \frac{1}{m\ell^2} (-k_p x_1 - k_d x_2 - mg\ell [\sin(x_1 + q_d) - \sin(q_d)]) \end{bmatrix}$$

We see that

$$V(x_1, x_2) = \frac{m\ell^2}{2} x_2^2 + \frac{k_p}{2} x_1^2 + mg\ell [\cos(q_d) - \cos(x_1 + q_d) - \sin(q_d)x_1]$$

$$\dot{V}(x_1, x_2) = -k_d x_2^2.$$

Define $M := \{x_2 = 0\} \cap \{x_1 \in \mathbb{R}\}$. GAS follows since $x_1 = 0$ is the only solution of

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{1}{m\ell^2} (-k_p x_1 - mg\ell [\sin(x_1 + q_d) - \sin(q_d)]) \end{bmatrix}$$

Example: pendulum

Adaptive set-point control: Consider the pendulum equation with an adaptive PD-plus-gravity-compensation controller, *i.e.*

$$\begin{aligned} m\ell^2\ddot{q} + mg\ell \sin(q) &= -k_p\tilde{q} - k_d\dot{q} + \hat{m}(t)g\ell \sin(q) \\ \dot{\hat{m}} &= -g\ell \sin(q)\dot{q} \end{aligned}$$

The state-space representation is

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ \frac{1}{m\ell^2} (-k_p x_1 - k_d x_2 - [m - \hat{m}(t)]g\ell \sin(x_1 + q_d)) \end{bmatrix}$$

We see that

$$\begin{aligned} V(x_1, x_2) &= \frac{m\ell^2}{2}x_2^2 + \frac{k_p}{2}x_1^2 + \frac{1}{2}(\hat{m}(t) - m)^2 \\ \dot{V}(x_1, x_2) &= -k_d x_2^2. \end{aligned}$$

Define $M := \{x_2 = 0\} \cap \{x_1 \in \mathbb{R}\} \cap \{(\hat{m} - m) \in \mathbb{R}\}$.

Example: pendulum

Adaptive set-point control: Consider the pendulum equation with an adaptive PD-plus-gravity-compensation controller, *i.e.*

$$\begin{aligned} m\ell^2\ddot{q} + mgl\sin(q) &= -k_p\tilde{q} - k_d\dot{q} + \hat{m}(t)g\ell\sin(q) \\ \dot{\hat{m}} &= -g\ell\sin(q)\dot{q} \end{aligned}$$

The state-space representation is

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ \frac{1}{m\ell^2} (-k_px_1 - k_dx_2 - [m - \hat{m}(t)]g\ell\sin(x_1 + q_d)) \end{bmatrix}$$

Define $M := \{x_2 = 0\} \cap \{x_1 \in \mathbb{R}\} \cap \{(\hat{m} - m) \in \mathbb{R}\}$.

The point $x_1 = 0, \hat{m} = m \in M$ **is not** the only solution of

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{1}{m\ell^2} (-k_px_1 - [m - \hat{m}(t)]g\ell\sin(x_1 + q_d)) \end{bmatrix}$$

For example: take $q_d = \pi/4, x_1 = \pi/4$. Then, $\hat{m}(t) = \frac{k_p\pi}{4g\ell} + m$.

Time-varying periodic systems

Extensions of [6, Theorem 4], to the case of non-autonomous periodic systems, have also been published; the first is probably due to N. N. Krasovskii:

(Cited from [22, Chapter 3, p. 66-67])

[...] we consider the more general case in which the equations

$$(14.1) \quad \frac{dx_i}{dt} = X_i(x_1, \dots, x_n, t) \quad (i = 1, \dots, n)$$

of [the] perturbed motion are such that the right members $X_i(x, t)$ are periodic functions of the time t with period ϑ , or do not depend explicitly on the time t . We further assume that the functions are defined in the region

$$(14.2) \quad \|x\| < H, \quad -\infty < t < \infty \quad (H = \text{const. or } H = \infty)$$

•

Time-varying periodic systems

THEOREM 14.1. *Suppose the equations of perturbed motion (14.1) enjoy the properties that*

- (i) *there exists a function $v(x, t)$ which is periodic in the time t with period ϑ or does not depend explicitly on the time;*
- (ii) *$v(x, t)$ is positive definite;*
- (iii) *$v(x, t)$ admits an infinitely small upper bound in the region (14.2);*
- (iv)
$$\sup (v \text{ in the region } \|x\| \leq H_0, \quad 0 \leq t < \vartheta) < \inf (v \text{ for } \|x\| = H_1) \quad (H_0 < H_1 < H);$$
- (v) *$dv/dt \leq 0$ in the region (14.2);*
- (vi) *the set M of points at which the derivative dv/dt is zero contains no nontrivial half trajectory $x(x_0, t, t), (0 < t < \infty)$, of the system (14.1)*

Under these conditions, the null solution $x = 0$ is asymptotically stable and the region $\|x\| \leq H_0$, lies in the region of attraction of the point $x = 0$.

Time-varying periodic systems

Modern formulations of the latter can be found for instance in [65, p. 179]; specifically, [65, Theorem 5.3.79], which is called by the author “Krasovskii-La Salle’s theorem”, corresponds to [22, Theorem 14.1] given above, to the case when $H = \infty$, *i.e.* the case of global asymptotic stability:

(Cited from [65, p. 179])

79 Theorem (Krasovskii-LaSalle) Suppose that the system (5.1.1) $[\dot{\mathbf{x}}(t) = \mathbf{f}[t, \mathbf{x}(t)], t \geq 0]$ is periodic. Suppose that there exists a C^1 function $V : \mathbb{R}_+ \times \mathbb{R}^n \rightarrow \mathbb{R}$ having the same period as the system such that (i) V is a pdf [positive definite] and is radially unbounded, and (ii)

$$\mathbf{80} \quad \dot{V}(t, \mathbf{x}) \leq 0, \quad \forall t \geq 0, \quad \forall \mathbf{x} \in \mathbb{R}^n.$$

Define

$$\mathbf{81} \quad R = \{\mathbf{x} \in \mathbb{R}^n : \exists t \geq 0 \text{ such that } \dot{V}(t, \mathbf{x}) = 0\},$$

and suppose R does not contain any trajectories of the system other than the trivial trajectory. Then the equilibrium $\mathbf{0}$ is globally uniformly asymptotically stable. ●

Uniform Stability

Uniform stability is defined as follows – see the equivalent definitions given in [65, p. 137], [54, p. 7], [2, p. 143], [20, Definition 4.4., p. 149]:

Definition 11 (Uniform stability) *The origin of the system (1) is said to be **uniformly stable** if for each $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ such that*

$$|x_o| \leq \delta \implies |x(t; t_o, x_o)| \leq \varepsilon \quad (\#)$$

for all $t \geq t_o$ and all $t_o \geq 0$. •

Different authors attribute this property to Persidskĭi: Antosiewicz, in [2], attributes it to [51] while Rouché *et al*, in [54], attribute it to [49]. Persidskĭi himself, in [51] refers to [49]:

Uniform Stability

(Cited and translated from [49])

Assume that $x_s = f_s(t, t_1)$, ($s = 1, 2, \dots, n$) is a system of continuous functions that satisfies the following system of differential equations of perturbed motion

$$\frac{dx_s}{dt} = y_s(x_1, x_2, \dots, x_n, t) \quad (s = 1, 2, \dots, n) \quad (1)$$

and which takes, for $t = t_1$ corresponding values ε_s [i.e., with initial conditions t_1 and $x_s(t_1) = \varepsilon_s$].

If the functions $x_s = f_s(t, t_1)$ are such that for any arbitrarily small number $H > 0$ there exists a number $h > 0$ such that for all values $t \geq t_1$ we will have

$$(x_1^2 + x_2^2 + \dots + x_n^2) \leq H, \quad (2)$$

if

$$(\varepsilon_1^2 + \varepsilon_2^2 + \dots + \varepsilon_n^2) \leq h \quad (3)$$

then the unperturbed motion will be called stable. [...]

Uniform Stability

If the functions $x_s = f_s(t, t_1)$ are such that for any arbitrarily small number $H > 0$ there exists a number $h > 0$ such that for all values $t \geq t_1$ we will have

$$(x_1^2 + x_2^2 + \cdots + x_n^2) \leq H, \quad (2)$$

if

$$(\varepsilon_1^2 + \varepsilon_2^2 + \cdots + \varepsilon_n^2) \leq h \quad (3)$$

[...] In general, the number h is a function of t_1 and H . In the case when for all values $t_1 \geq t_0$ there exists a number h , which is independent of t_1 , we will call the unperturbed motion uniformly stable. ●

- Persidskii does not assume the usual local Lipschitz, uniform in t , property on the function $y_s(\cdots, t)$ but that there exist continuous positive functions $A_s(t)$ such that

$$|y_s(x_1, x_2, \dots, x_n, t) - y_s(x'_1, x'_2, \dots, x'_n, t)| \leq A_s(t)(|x_1 - x'_1| + \cdots + |x_n - x'_n|)$$

and y_s are continuous.

Uniform Stability

Starting with W. Hahn, the notation from Definition 8 has been used. More precisely, we have the following:

(Cited from [14, p. 62])

THEOREM 17.1: The equilibrium of differential equation (2.7) $\left[\begin{array}{l} \dot{x} = f(x, t), \\ f(0, t) = 0, f \in E \end{array} \right]$ is uniformly stable if and only if there exists a function $\rho(r)$ with the following properties:

- (a) $\rho(r)$ is defined, continuous, and monotonically increasing in an interval $0 \leq r \leq r_1$;
- (b) $\rho(0) = 0$; the function ρ , therefore, belongs to the class K ;
- (c) the inequality

$$|p(t, x_o, t_o)| \leq \rho(|x_o|)$$

is valid for $|x_o| < r_1$.



Uniform Stability

Hahn attributes the following result to [50]:

(Cited from [14]) THEOREM 17.6 If there exists a positive definite decrescent Liapunov function v such that its total derivative \dot{v} for (2.7) is negative semi-definite, then the equilibrium is stable. •

The sufficiency theorem cited above is also attributed by Rouche *et al* [54] to K. P. Persidskii –[50] while Antosiewicz [2] cites [51]. Indeed, Persidskii gives in, [50] and for the first time, necessary and sufficient conditions for uniform stability. That is, Persidskii's original statement is more general than that contained in [14, THEOREM 17.6] however, its precise formulation requires the introduction of other definitions[†]

[†]It is worth pointing out that among these definitions, Persidskii uses the terminology “class \mathcal{L} ” function however, Persidskii's definition is different from “Hahn's” definition of class \mathcal{L} function, which is commonly used nowadays. that we do not detail here.

Uniform global stability

Thus, uniform global stability is defined as follows:

(Cited from [24, p. 490])

Definition A.4 The equilibrium point $x = 0$ of $[\dot{x} = f(x, t)]$ is

- uniformly stable, if there exists a class \mathcal{K} function $\gamma(\cdot)$ and a positive constant c independent of t_o , such that

$$|x(t)| \leq \gamma(|x(t_o)|), \quad \forall t \geq t_o, \quad \forall x(t_o) \mid |x(t_o)| < c; \quad (A.3)$$

- globally uniformly stable, if (A.3) is satisfied with $\gamma \in \mathcal{K}_\infty$ for any initial state $x(t_o)$.

•

This is equivalent to Hahn's uniform stability *in the whole*:

(Cited from [14, p. 62])

DEFINITION 17.2: The equilibrium of (2.7) $[\dot{x} = f(x, t), f(0, t) = 0, f \in E]$ is said to be uniformly stable in the whole if the assumption of Theorem 17.1 are satisfied for every arbitrarily large r_1 .

•

Uniform Asymptotic Stability

Uniform asymptotic stability appears, **implicitly**, in many articles of I. G. Malkin between 1940 and 1955 in the context of stability with respect to *constantly-acting disturbances*. Hahn attributes the following to Malkin:

(Cited from [14])

DEFINITION 17.4 (Malkin [20]): *The equilibrium of (2.7) is called uniformly asymptotically stable if*

1. *the equilibrium is uniformly stable*
2. *for every $\epsilon > 0$ a number $\tau = \tau(\epsilon)$ depending only on ϵ , but not on the initial instant t_0 can be determined such that the inequality*

$$|p(t, x_0, t_0)| < \epsilon \quad (t > t_0 + \tau)$$

holds, provided x_0 belongs to a spherical domain \mathfrak{R}_η whose radius η is independent of ϵ . •

- Hahn's reference "Malkin [20]" corresponds to the paper [39], on the converse Lyapunov theorem for uniform asymptotic stability and on stability with respect to constantly-acting perturbations.

Uniform Attractivity

- The second part of [14, Definition 17.4] is often referred to as *uniform attractivity* as a qualifier for the equilibrium; the following interesting characterisation is seemingly due to Hahn:

(Cited from [14, p. 64]) Necessary and sufficient for the second condition of Definition 17.4 is the existence of a function $\sigma(r)$ with the following properties:

- (a) $\sigma(r)$ is defined, continuous, and monotonically decreasing, for all $r \geq 0$,
- (b) $\lim_{r \rightarrow \infty} \sigma(r) = 0$,
- (c) provided the initial points belong to a fixed spherical domain \mathfrak{R}_η , the relation

$$|p(t, x_o, t_o)| \leq \sigma(t - t_o) \quad (17.6)$$

holds.



Uniform Attractivity

(Cited from [14, p. 64])

THEOREM 17.4 (Hahn): *Necessary and sufficient for uniform asymptotic stability of the equilibrium is the existence of two functions $\kappa(r)$ and $\vartheta(r)$ with the following properties:*

- (a) $\kappa(r)$ satisfies assumptions (a) and (b) of Theorem 17.1,
- (b) $\vartheta(r)$ satisfies the corresponding assumptions of Theorem 17.3;
- (c) in addition, the inequality

$$|p(t, x_o, t_o)| \leq \kappa(|x_o|)\vartheta(t-t_o) \quad (17.7)$$

holds, provided that the initial points x_o belong to a fixed spherical domain \mathfrak{R}_η .



Uniform Asymptotic Stability in the Whole

(Cited and translated from [7, p. 346])

We call the solution $x_1 = \dots = x_n = 0$ of system (1)

$$\left[\frac{dx_i}{dt} = X_i(x_1, \dots, x_n, t) \quad (i = 1, \dots, n) \quad (1) \right]$$

[is] uniformly [asymptotically] stable in the whole, if for any numbers $R_1 > 0$ and $R_2 > 0$ one can find a number $T(R_1, R_2)$, depending continuously only on R_1 and R_2 , such that, any solution $x_i(x_{10}, \dots, x_{n0}, \tau_0, t)$ ($i = 1, \dots, n$) with initial values for $t = \tau_0 \geq t_0$ laying in the region

$$x_{10}^2 + \dots + x_{n0}^2 \leq R_1^2,$$

satisfies inequality

$$x_1^2 + \dots + x_n^2 < R_2^2 \quad \text{for } \tau_0 + T(R_1, R_2)$$

and at same time for any number $R_1 > 0$ there exists a number $R_2 = F(R_1)$, depending continuously only on R_1 , such that any trajectory starting from the interior of a sphere of radius R_1 does not escape from a sphere of radius R_2 as time passes.

Uniform Asymptotic Stability in the Whole

- Note that the term “asymptotically” is omitted by the authors
- The first part of the definition corresponds to uniform global attractivity while the second part corresponds to uniform boundedness of solutions hence, the constants R_1 and R_2 are not the same in each of the two parts of the definition
- Furthermore, from the rigorous viewpoint that characterises modern literature, the definition above actually does not explicitly include uniform *stability*
- In addition, it is not made explicit whether the function F is radially unbounded.

Uniform Asymptotic Stability in the Whole

Barbashin and Krasovskii's formulation is closely followed (and formalised) by Hahn –cf. [14, Definition 17.5]:

(Cited from [14, p. 64])

DEFINITION 17.5 (Barbashin and Krasovskii [2]): The equilibrium of differential equation (2.7) is said to be uniformly asymptotically stable in the whole, if the following two conditions are satisfied:

- (a) The equilibrium is uniformly stable in the whole;
- (b) for any two numbers $\delta_1 > 0$ and $\delta_2 > 0$ there exists a number $\tau(\delta_1, \delta_2)$ such that

$$|p(t, x_o, t_o)| < \delta_2$$

if $t \geq t_o + \tau(\delta_1, \delta_2)$ and $|x_o| < \delta_1$. •

Two definitions of UGAS

$$\dot{x} = f(t, x) . \quad (6)$$

Definition 12 (*[54, p. 10]^a, [42, p. 356], [3, Definition 3.6, p. 80], [65, Definition 5.1.38, p. 143]*) *The origin of (6) is said to be uniformly globally asymptotically stable (UGAS) if it is uniformly stable –cf. Definition 11 and uniformly globally attractive, i.e. if for any $r > 0$ and $\sigma > 0$ there exists $T(\sigma, r) > 0$ such that, for all $t_o \geq 0$,*

$$|x_o| \leq r \implies |x(t; t_o, x_o)| \leq \sigma \quad \forall t \geq t_o + T . \quad \bullet$$

Definition 13 (*[14, Defin. 17.5, p. 64], [20, p. 150], [17, Defin. 2.7, p. 38]*) *The origin of the system (6) is uniformly globally asymptotically stable if*

1. *it is uniformly globally stable, i.e. there exists $\gamma \in \mathcal{K}_\infty$ such that*

$$|x(t)| \leq \gamma(|x_o|) \quad (7)$$

2. *it is uniformly globally attractive.* •

^aThe authors attribute this definition to [6] which concerns only autonomous systems.

Lyapunov Stability

[Equivalent characterisation]

Definition 13 (stability) *The origin of $\dot{x} = f(t, x)$ is said to be*

- *stable if for each pair (t_o, ε) , s.t. $t_o \geq 0, \varepsilon > 0$ there exists $\delta(t_o, \varepsilon) > 0$ such that*

$$|x_o| \leq \delta \implies |x(t; t_o, x_o)| \leq \varepsilon \quad \forall t \geq t_o \geq 0. \quad (8)$$

- *uniformly stable if for each $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ such that (8) holds*
- *uniformly globally stable if $\delta(\varepsilon)$ can be chosen such that $\delta(\varepsilon) \rightarrow \infty$ as $\varepsilon \rightarrow \infty$*

Definition 14 (Uniform/global/stability) *The origin of $\dot{x} = f(t, x)$ is:*

- *stable if and only if there exist $r > 0$ and, for each $t_o \geq 0, \gamma \in \mathcal{K}$ such that*

$$|x_o| \leq r \implies |x(t, t_o, x_o)| \leq \gamma(|x_o|) \quad \forall t \geq t_o, \quad (9)$$

- *uniformly stable if and only if there exist $r > 0$ and $\gamma \in \mathcal{K}$ such that (9) holds;*
- *uniformly **globally** stable if and only if there exists $\gamma \in \mathcal{K}_\infty$ s.t.*

$$|x(t, t_o, x_o)| \leq \gamma(|x_o|) \quad \forall t \geq t_o \geq 0, \quad x_o \in \mathbb{R}^n$$

Lyapunov's Asymptotic Stability

Definition 15 (Uniform/global/attractivity) *The origin of the system $\dot{x} = f(t, x)$ is said to be attractive if there exists $r > 0$ and, for each $t_o \geq 0$ and $\sigma > 0$ there exists $T > 0$ such that*

$$|x_o| \leq r \implies |x(t, t_o, x_o)| \leq \sigma \quad \forall t \geq t_o + T . \quad (10)$$

*Moreover, it is **uniformly** attractive if T is **independent of t_o** that is, if there exists $r > 0$ and, for each $\sigma > 0$ there exists $T > 0$ such that (10) holds.*

*Finally, it is said to be uniformly globally attractive if **for each** $r > 0$ and $\sigma > 0$ there exists $T > 0$ such that (10) holds.* •

Definition 16 (Uniform Global Asymptotic Stability) *The origin of the system $\dot{x} = f(t, x)$ is said to be uniformly globally asymptotically stable if it is*

- *(locally) uniformly stable* $|x(t)| \leq \gamma(|x_o|), \gamma \in \mathcal{K};$
- *the solutions are uniformly globally bounded* $|x(t)| \leq R(r)$ for any $r > 0;$
- *the origin is uniformly globally attractive.* •

Uniformity implies Robustness

Definition 17 (Total stability^a) *The origin of $\dot{x} = f(t, x, 0)$, is said to be totally stable if, for the system $\dot{x} = f(t, x, u)$ small bounded inputs $u(t, x)$ and small initial conditions $x_o = x(t_o)$, yield small state trajectories for all $t \geq t_o$, i.e., if for each $\varepsilon > 0$ there exists $\delta > 0$ such that*

$$\max \{ \|x_o\|, \|u\|_\infty \} \leq \delta \quad \implies \quad \|x(t, t_o, x_o, u)\| \leq \varepsilon \quad \forall t \geq t_o \geq 0. \quad (11)$$

•

(Non)-uniform, even exponential, stability does not guarantee robustness in the sense above –see[5, Ch. 3, sec. 5]

$$\dot{x} = -(a - \sin \ln(t + 1) - \cos \ln(t + 1))x,$$

where $1 < a < 1 + (1/2)e^{-\pi}$. It should be noted that even though the solutions of this system are exponentially convergent, the origin of the system is not uniformly stable.

^aThe definition provided here, is a modern version of total stability, more suitable for the purposes of this paper. The notion of total stability was originally introduced in [38].

Example 3 (Panteley, Teel '98) Consider the system $\dot{x} = f(t, x)$ with

$$f(t, x) = \begin{cases} -a(t)\operatorname{sgn}(x) & \text{if } |x| \geq a(t) \\ -x & \text{if } |x| \leq a(t) \end{cases} \quad (12)$$

and $a(t) = \frac{1}{t+1}$. This system has the following properties:

1. The function $f(t, x)$ is globally Lipschitz in x , uniformly in t and the system is UGS with linear gain equal to one.
2. For each $r > 0$ and $t_o \geq 0$ there exist strictly positive constants κ and λ such that for all $t \geq t_o$ and $|x(t_o)| \leq r$

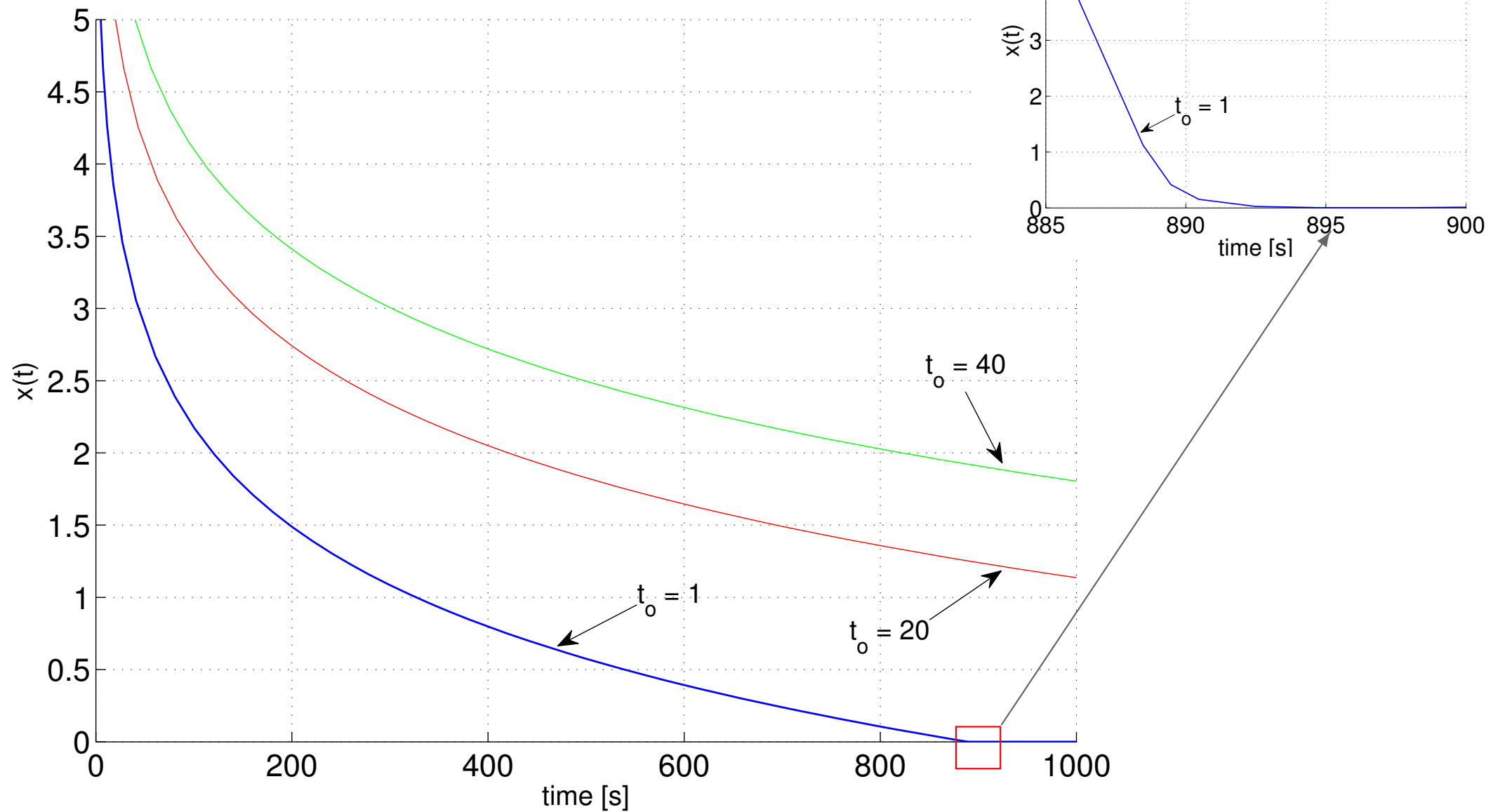
$$|x(t)| \leq \kappa |x(t_o)| e^{-\lambda(t-t_o)} \quad (13)$$

3. The origin is not totally stable. Indeed, for any $\delta \ll 1$, there exist $(t_o, x_o) \in \times B_r$, with $\delta > x_o > a(t_o)$, s.t., the solution of

$$\dot{x} = [-a(t) + \delta]\operatorname{sgn}(x)$$

grows unboundedly. •

$$\dot{x}(t, t_o, x_o) = \begin{cases} -a(t)\text{sgn}(x(t)) & \text{if } |x(t)| \geq a(t) \\ -x & \text{if } |x(t)| \leq a(t) \end{cases}$$



Conclusions

- A number of stability definitions have been proposed in the literature. Sometimes, misconceptions of stability have resulted from wrong translations and mis-use of terminology.
- Only via a careful book-keeping and a “return to the sources” clarifications may be brought up.
- Lagrange-Dirichlet’s stability is reminiscent of stability with respect to part of coordinates;
- Lyapunov’s stability is a much broader concept than usually considered;
- Two definitions of uniform global asymptotic stability have been used; only one of which yields converse results;
- Asymptotic stability in the large and global asymptotic stability are **not** synonyms;

**“Des lecteurs attentifs, qui se communiquent leurs pensées,
vont toujours plus loin que l’auteur”[†]**

—Voltaire, 1763.

[†]Attentive readers, which communicate their thoughts always go beyond the author

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