

# Output Feedback Regulation and Reference Tracking for Constrained Linear Systems Using Invariant Sets

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# Outline

- 1 Controlled Invariant Polyhedral Sets
- 2 Output-Feedback Controlled-Invariant Polyhedra
- 3 Dynamic Output Feedback Control
- 4 Constant Reference Tracking with Disturbance Rejection
- 5 Conclusions

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# Linear Systems Subject to Constraints

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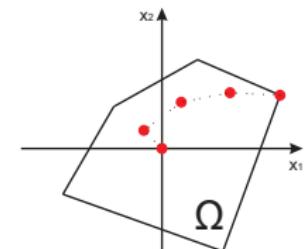
- (1) drive the state of a system to an equilibrium point;
- (2) satisfy at the same time a set of constraints.

Linear constraints on state, control, and/or output variables:  
 $\Rightarrow$  controlled invariance of convex polyhedra  $\Rightarrow Gx \leq \rho$ .

- Consider the linear, time-invariant, discrete-time system:

$$x(k+1) = Ax(k) + Bu(k) + Ed(k), \quad (1)$$

where  $k \in \mathbb{N}$ ,  $x(k) \in \Omega_x \subset \mathbb{R}^n$ ,  $u(k) \in \mathfrak{U} \subset \mathbb{R}^m$ ,  $d(k) \in \mathfrak{D} \subset \mathbb{R}^r$ .



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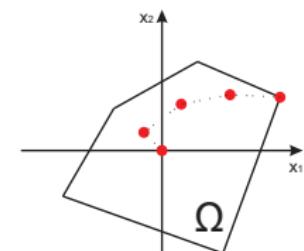
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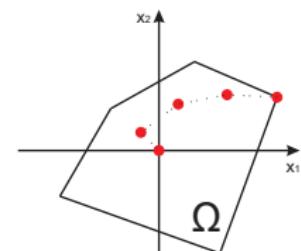
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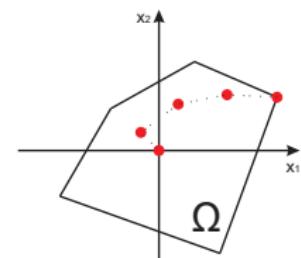
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Definition: A set  $\Omega \subseteq \Omega_x$  is controlled-invariant if

The set  $\Omega \subseteq \Omega_x$  is said to be controlled-invariant with contraction rate  $\lambda$ ,  $0 \leq \lambda < 1$ , w.r.t. system (1) if  $\forall x \in \Omega, \exists u \in \mathfrak{U}: Ax + Bu + Ed \in \lambda\Omega, \forall d \in \mathfrak{D}$ .  
 $\Rightarrow$  a state feedback control enforces the state trajectory to remain in  $\Omega$ .



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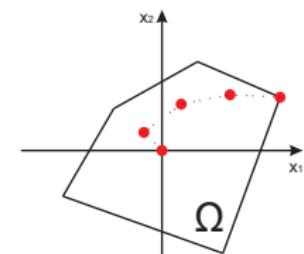
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## Definition (Controlled Invariance)

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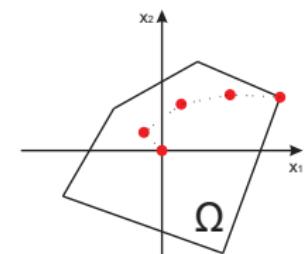
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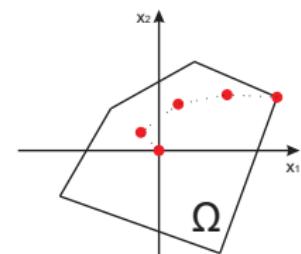
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# Controlled Invariance of Polyhedral Sets

Many works on state feedback control schemes:

Blanchini, 1994: vertex-based necessary and sufficient conditions + computation of the maximal set + piecewise-linear control;

Dórea and Hennet, 1999: hiperplane-based necessary and sufficient conditions + computation of the maximal set;

Rakovic and Baric, 2010: parametrized controlled invariant sets;

Gutman and Cwikel, 1984: vertex control;

Nguyen, Gutman, Olaru and Hovd, 2013: interpolation-based control.

# Controlled Invariance of Polyhedral Sets

Not so many on output feedback control schemes:

Artstein and Rakovic, 2011: set-dynamics approach;

Nguyen, Gutman, Olaru and Hovd, 2011: interpolation-based approach using non-minimal state space model;

Lee and Kouvaritakis, 2001: observer-based MPC technique;

Mayne, Rakovic, Findeisen and Allgower, 2006:  
observer-based MPC techniques;

Dórea, 2009: Output Feedback Controlled Invariant (OFCI) of polyhedral sets.

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# Output Feedback Control

Control problem:

Compute a control sequence  $u(y(k)) \in \mathfrak{U}$  such that  $x(k) \in \Omega_x$ ,  
 $\forall k \in \mathbb{N}$ .

Possible solution:

Compute an Output-Feedback Controlled-Invariant (OFCI)  
polyhedral set contained in the set defined by the constraints.  
⇒ no method is available for such a computation.

However...

Conditions were established under which a controlled invariant  
set can also be made invariant by *static* output feedback.<sup>a</sup>

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<sup>a</sup>Dórea, C.E.T. (2009). Output-Feedback Controlled Invariant Polyhedra for Constrained Linear Systems. In *Proceedings of the 48th IEEE Conference on Decision and Control*, 5317 – 5322.

# Output Feedback Controlled Invariant Sets

- Consider the linear, time-invariant, discrete-time system:

$$x(k+1) = Ax(k) + Bu(k) + Ed(k) \quad (2)$$

$$y(k) = Cx(k) + \eta(k) \quad (3)$$

Convex polyhedra

$\Omega = \{x : Gx \leq \bar{1}\}$ ,  $\mathfrak{U} = \{u : Uu \leq \bar{1}\}$ ,  $\mathfrak{D} = \{d : Dd \leq \bar{1}\}$ , and  $\mathcal{N} = \{\eta : N\eta \leq \bar{1}\}$ .  $G \in \mathbb{R}^{g \times n}$ ,  $U \in \mathbb{R}^{v \times m}$ ,  $D \in \mathbb{R}^{s \times r}$ ,  $N \in \mathbb{R}^{q \times p}$ .

Admissible outputs

$$\mathcal{Y}(\Omega) = \{y : y = Cx + \eta \text{ for } x \in \Omega, \eta \in \mathcal{N}\} = C\Omega \oplus \mathcal{N}.$$

Possible states associated to a single measurement  $y$

$$\mathfrak{C}(y) = \{x : Cx = y - \eta, \text{ for } \eta \in \mathcal{N}\}.$$

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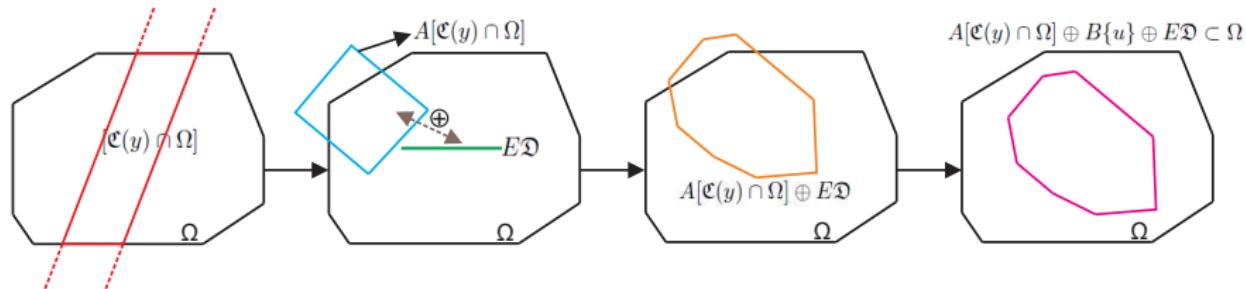
# Output Feedback Controlled Invariant Sets

## Definition (OFCI set)

The set  $\Omega \subset \mathbb{R}^n$  is said to be Output-Feedback Controlled-Invariant (OFCI) with contraction rate  $\lambda$ ,  $0 \leq \lambda < 1$ , w.r.t. system (2)-(3) if  $\forall y \in \mathcal{Y}(\Omega)$ ,  $\exists u \in \mathfrak{U}$  such that  $Ax + Bu + Ed \in \lambda\Omega$ ,  $\forall d \in \mathfrak{D}$  and  $\forall x \in \Omega$ ,  $\eta \in \mathcal{N}$  such that  $Cx = y - \eta$ .

## In geometric terms

$\Omega$  is OFCI if  $\forall y \in \mathcal{Y}(\Omega)$ ,  $\exists u \in \mathfrak{U} : A[\mathfrak{C}(y) \cap \Omega] \oplus Bu \oplus E\mathfrak{D} \subset \lambda\Omega$ .



# Output Feedback Controlled Invariance

$\Omega$  is OFCI  $\Rightarrow \forall y \in \mathcal{Y}(\Omega),$

$$\exists u \in \mathfrak{U} : G(Ax + Bu + Ed) \leq \lambda \bar{1}, \quad Uu \leq \bar{1}$$

$$\forall x, \eta, d : Gx \leq \bar{1}, \quad y - Cx = \eta, \quad N\eta \leq \bar{1}, \quad Dd \leq \bar{1}.$$

worst case  $x:$

$$\phi_j(y) = \max_x G_j Ax, \quad j = 1, \dots, g$$

$$\text{s.t. } Gx \leq \bar{1}, \quad -NCx \leq -Ny + \bar{1}.$$

worst case  $d:$

$$\delta_j = \max_d G_j Ed, \quad j = 1, \dots, g$$

$$\text{s.t. } Dd \leq \bar{1}.$$

For invariance:

$$\forall y \in \mathcal{Y}(\Omega), \quad \exists u \in \mathfrak{U} : \begin{bmatrix} \phi(y) \\ \bar{0} \end{bmatrix} + \begin{bmatrix} GB \\ U \end{bmatrix} u \leq \begin{bmatrix} \lambda \bar{1} - \delta \\ \bar{1} \end{bmatrix}.$$

$\Rightarrow$  crucial inequalities  $\Rightarrow$  once satisfied we can assure the invariance of the polyhedron  $\Omega.$

# Conditions for Polyhedral OFCI

Projection over the output space:

$\left\{ \begin{bmatrix} T_i & W_i \end{bmatrix}^T, i = 1, \dots, n_r \right\}$  form a *minimal generating set* of  $\Pi$ .

Then, OFCI of  $\Omega$  amounts to:

$\forall y \in \mathcal{Y}(\Omega), \forall i = 1, \dots, n_r:$

$$\begin{bmatrix} T_i & W_i \end{bmatrix} \begin{bmatrix} \phi(y) \\ 0 \end{bmatrix} \leq \begin{bmatrix} T_i & W_i \end{bmatrix} \begin{bmatrix} \lambda \bar{1} - \delta \\ \bar{1} \end{bmatrix}$$

# Conditions for Polyhedral OFCI

Theorem:

$\Omega$  is OFCI with contraction rate  $\lambda$  if and only if,  $\forall i = 1, \dots, n_r$ :

$$\sum_{j=1}^g T_{ij} G_j A \Xi^j \leq \left[ \sum_{j=1}^g T_{ij} (\lambda - \delta_j) \right] + W_i \bar{1}$$
$$\forall y, \Xi^j, j = 1, 2, \dots, g : G \Xi^j \leq \bar{1}, -N C \Xi^j \leq -Ny + \bar{1}$$

# Conditions for Polyhedral OFCI

OFCI test via Linear Programming:

$$\epsilon_i = \max_{y, \Xi^j} \sum_{j=1}^g T_{ij} G_j A \Xi^j$$

$$\text{s.t. } G \Xi^j \leq \bar{1}, \quad -N C \Xi^j \leq -Ny + \bar{1}$$

$\Omega$  is OFCI with contraction rate  $\lambda$  if and only if,  $\forall i = 1, \dots, n_r$ :

$$\epsilon_i + \sum_{j=1}^g T_{ij} \delta_j - W_i \bar{1} \leq \left( \sum_{j=1}^g T_{ij} \right) \lambda$$

# Online Control Strategy

Given an OFCI polyhedron  $\Omega$  and assuming  $x(k) \in \Omega$ , compute  $u(k) \in \mathfrak{U}$  such that all admissible  $x(k+1)$  (consistent with the measurement  $y(k)$ ) belong to the smallest ball around the origin:

$$\mathcal{B}(\varepsilon) = \{x : -\varepsilon \leq x \leq \varepsilon\} = \{x : Hx \leq \varepsilon \bar{1}\}.$$

$\forall y \in \mathcal{Y}(\Omega)$ , compute  $u \in \mathfrak{U}$  such that:

$$H(Ax + Bu + Ed) \leq \varepsilon \bar{1}$$

$$\forall x, \eta, d : Gx \leq \bar{1}, y - Cx = \eta, N\eta \leq \bar{1}, Dd \leq \bar{1}.$$

worst case  $x$ :

$$\varphi_j(y) = \max_x H_j Ax, \quad j = 1, \dots, 2n$$

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worst case  $d$ :

$$\gamma_j = \max_d H_j Ed, \quad j = 1, \dots, 2n$$

$$\text{s.t. } Dd \leq \bar{1}.$$

Control action (computed *online*)

$$u(y(k)) = \arg \min_{u, \varepsilon} \varepsilon$$

$$\text{s.t. } \begin{bmatrix} GB & \bar{0} \\ U & \bar{0} \\ \hline HB & -1 \end{bmatrix} \begin{bmatrix} u \\ \varepsilon \end{bmatrix} \leq \begin{bmatrix} \bar{1} - \phi(y(k)) - \delta \\ \bar{1} \\ -\varphi(y(k)) - \gamma \end{bmatrix}.$$

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# State Estimators

- **Main goal:** reduce the uncertainty on  $x(k)$  given  $y(k)$ .
- Two possible approaches:

## Set-Valued Observers Based on Zadeh Membership Techniques

At each time instant, the set of states which could generate the measured output is computed *online* and an optimal state is selected <sup>a</sup>  $\Rightarrow$  heavy online computation

<sup>a</sup>Shamma, J. S. and Tu, K. Y. (1999). Set-Valued Observers and Optimal Disturbance Rejection, *IEEE Transactions on Automatic Control* 44(2): 253-264.

## Set-Invariant Observers Based on Invariant polyhedral sets

Full-order state observers with limitation of the estimation error. <sup>b</sup>  $\Rightarrow$  *conditioned-invariant* sets.

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# Conditioned Invariant Sets

- "Dual" of controlled-invariant sets.
- Possibly nonlinear output injection.

Conditioned Invariance: the decomposed form

$$\forall y \in \mathcal{Y}(\Omega), \exists v(.) : A[\mathcal{C}(y) \cap \Omega] \oplus v(.) \oplus E\mathcal{D} \subset \lambda\Omega.$$

Relationship with Output Feedback Controlled Invariance

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A dynamic output feedback compensator structure allows the construction of a polyhedral set satisfying this necessary condition.

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$\Omega$  is OFCI only if it is simultaneously controlled invariant and conditioned invariant.

A dynamic output feedback compensator structure allows the construction of a polyhedral set satisfying this necessary condition.

# Conditioned Invariant Sets

- "Dual" of controlled-invariant sets.
- Possibly nonlinear output injection.

Conditioned Invariance - In geometric terms

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# Dynamic Output-Feedback Compensator

- Full-order, possibly nonlinear, compensator:

$$z(k+1) = v[z(k), y(k)], \quad u(k) = \kappa[z(k), y(k)]$$

$\Rightarrow$  system + compensator  $\Rightarrow$  extended state space formulation:

$$\begin{bmatrix} x(k+1) \\ z(k+1) \end{bmatrix} = \begin{bmatrix} A & \bar{0} \\ \bar{0} & \bar{0} \end{bmatrix} \begin{bmatrix} x(k) \\ z(k) \end{bmatrix} + \begin{bmatrix} B & \bar{0} \\ \bar{0} & I \end{bmatrix} \begin{bmatrix} u(k) \\ v(k) \end{bmatrix} + \begin{bmatrix} E \\ \bar{0} \end{bmatrix} d(k),$$

$$\begin{bmatrix} y(k) \\ z(k) \end{bmatrix} = \begin{bmatrix} C & \bar{0} \\ \bar{0} & I \end{bmatrix} \begin{bmatrix} x(k) \\ z(k) \end{bmatrix} + \begin{bmatrix} \eta(k) \\ \bar{0} \end{bmatrix}$$

or

$$\begin{aligned} \xi(k+1) &= \hat{A}\xi(k) + \hat{B}\omega(k) + \hat{E}d(k), \\ \zeta(k) &= \hat{C}\xi(k) + \hat{\eta}(k). \end{aligned}$$

# OFCI Polyhedron w.r.t. the Augmented System

Pair of polyhedral sets  $(\mathcal{V}, \mathcal{S})$

- $\mathcal{S} = \{x : G_s x \leq \bar{1}\} \subseteq \mathcal{V} = \{x : G_v x \leq \bar{1}\} \subseteq \Omega_x$ ;
- $\mathcal{S}$  is conditioned-invariant  $\lambda_s$ -contractive,  $\mathcal{V}$  is controlled-invariant  $\lambda_v$ -contractive.

Proposition

$$\widehat{\Omega} = \left\{ \begin{bmatrix} x \\ z \end{bmatrix} : \begin{bmatrix} G_v & \bar{0} \\ G_s & -G_s \end{bmatrix} \begin{bmatrix} x \\ z \end{bmatrix} \leq \begin{bmatrix} \bar{1} \\ \bar{1} \end{bmatrix} \right\} \Leftrightarrow \{\xi : \widehat{G}\xi \leq \bar{1}\}$$

is simultaneously controlled and conditioned-invariant.

Consequence:

- $x(k) \in \mathcal{V}, \forall k \Rightarrow G_v x(k) \leq \bar{1} \Rightarrow$  constraints satisfaction;
- $e(k) = x(k) - z(k) \in \mathcal{S}, \forall k \Rightarrow G_s[x(k) - z(k)] \leq \bar{1} \Rightarrow$  bounded “estimation error”.

## Set of Admissible Initial States

$$\begin{bmatrix} G_v & \bar{0} \\ G_s & -G_s \end{bmatrix} \begin{bmatrix} x(0) \\ z(0) \end{bmatrix} \leq \begin{bmatrix} \bar{1} \\ \bar{1} \end{bmatrix}$$

$$x(0) \in \mathcal{V} \cap (z(0) \oplus \mathcal{S})$$

### Consequences:

- The larger  $\mathcal{S}$ , the larger is the set of admissible initial states.
- Observer-based MPC strategies require small  $\mathcal{S}$  in general
- The OFCI approach is likely to result in much larger sets of admissible initial states.

# Proposed Control Strategy

Linear State Estimator:

$$z(k+1) = Az(k) + Bu(k) + L[y(k) + \hat{y}(k)], \quad \hat{y}(k) = Cz(k).$$

Dynamics of the estimation error  $e(k) = x(k) - z(k)$ :

$$e(k+1) = (A - LC)e(k) + \begin{bmatrix} E & -L \end{bmatrix} \begin{bmatrix} d(k) \\ \eta(k) \end{bmatrix} = A_e e(k) + E_e d_e(k).$$

$\lambda$ -contractive set w.r.t.  $e(t)$

$\mathcal{S}_m$   $\lambda_m$ -contractive and such that the pair  $(\mathcal{V}, \mathcal{S}_m)$  forms an OFCI polyhedron.

- If  $\mathcal{S}_m$  is conditioned-invariant, then, so is  $\alpha_m \mathcal{S}_m$ ,  $\forall \alpha_m > 1$  with a guaranteed contraction rate  $\lambda_m$ .
- Scale  $\mathcal{S}_m$  by the largest factor  $\alpha_m > 1$  such that  $(\mathcal{V}, \alpha_m \mathcal{S}_m)$  still forms an OFCI set  $\Rightarrow$  **enlargement of the set of admissible initial states.**

# Proposed Control Strategy

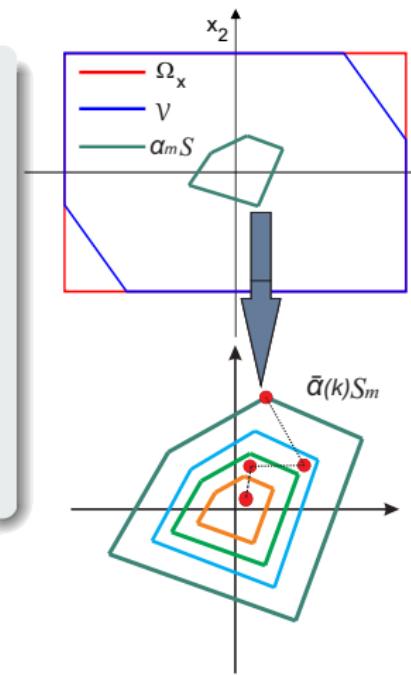
- Offline computation of the sequence of homothetic sets  $\bar{\alpha}(k)S_m$  visited by  $e(k)$  until reaching the smallest set  $S_m \Rightarrow$  **progressive reduction of the uncertainty on  $x(k)$**

$$u(k) = \arg \min_{u, \varepsilon} \varepsilon$$

s.t. 
$$\begin{bmatrix} GB & \bar{0} \\ U & \bar{0} \\ HB & -1 \end{bmatrix} \begin{bmatrix} u \\ \varepsilon \end{bmatrix} \leq \begin{bmatrix} \bar{1} - \phi(k) - \delta \\ \bar{1} \\ -\varphi(k) - \gamma \end{bmatrix}$$

$$\begin{cases} \phi_j(k) = \max_x G_{vj}Ax, & j = 1, \dots, g_v \\ \varphi_j(k) = \max_x H_jAx, & j = 1, \dots, 2n \end{cases}$$

s.t. 
$$\begin{bmatrix} G_v \\ G_s \end{bmatrix} x \leq \begin{bmatrix} \bar{1} \\ \bar{\alpha}(k)\bar{1} + G_s z(k) \end{bmatrix}, -NCx \leq -Ny(k) + \bar{1}$$



# Numerical Example

(Subramanian et al., 2017):

$$\begin{aligned}x(k+1) &= \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} x(k) + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u(k) + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} d(k) \\y(k) &= \begin{bmatrix} 1 & 1 \end{bmatrix} x(k) + \eta(k)\end{aligned}$$

- State and control constraints:

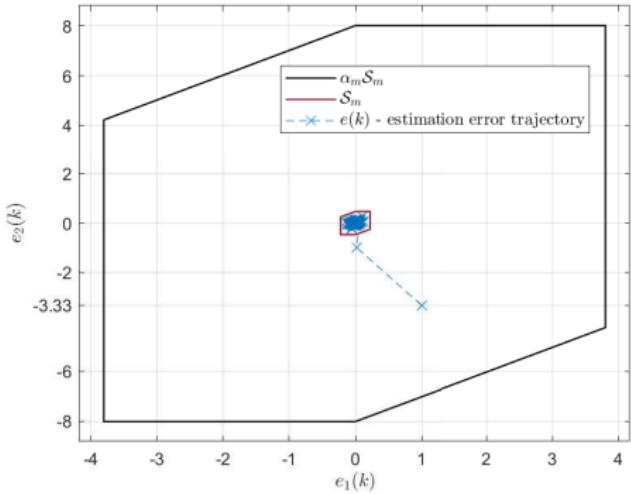
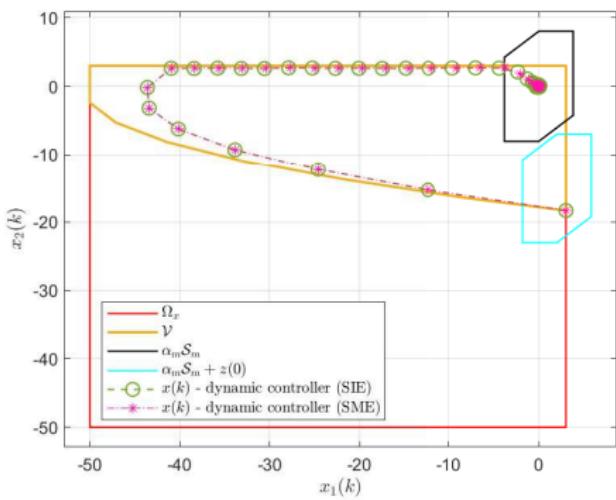
$$\Omega_x = \{x : -50 \leq x_i \leq 3, i = 1, 2.\} \text{ and } \mathfrak{U} = \{u : |u| \leq 3\}.$$

- Bounds for disturbances and measurement noise:

$$\mathfrak{D} = \{d : |d_i| \leq 0.1, i = 1, 2.\} \text{ and } \mathcal{N} = \{\eta : |\eta| \leq 0.1\}.$$

- $\mathcal{V}$  is the maximal controlled-invariant set (it is not OFCI) contained in  $\Omega_x$ , with contraction rate  $\lambda_v = 0.99$ .
- $\mathcal{S} = \alpha_m \mathcal{S}_m$ , where:
  - $\mathcal{S}_m$  is the mRPI set with contraction rate  $\lambda_m = 0.9$ ;
  - $\alpha_m = \max \alpha$  such that the pair  $(\mathcal{V}, \alpha_m \mathcal{S})$  is OFCI ( $\alpha_m = 17.1$ ).
- $\{\bar{\alpha}(k)\}_{k=0}^6 = \{17.1, 8.5263, 4.4651, 2.5414, 1.6301, 1.1985, 0.9940\}$
- $x(0) = [3 \quad -18.33]^T$  and  $z(0) = [2 \quad -15]^T$ .

# Numerical Example



The invariant set associated to the estimation error in (Subramanian et al., 2017) is given by  $\{e : |e_1| \leq 0.7, |e_2| \leq 1.4\}$

$\Downarrow$

Much smaller set of admissible initial states

- 1 Controlled Invariant Polyhedral Sets
- 2 Output-Feedback Controlled-Invariant Polyhedra
- 3 Dynamic Output Feedback Control
- 4 Constant Reference Tracking with Disturbance Rejection
- 5 Conclusions

# Constant Reference Tracking

Linear system with the same number of inputs and outputs:

$$x(k+1) = Ax(k) + Bu(k) + Ed(k), \quad (4)$$

$$y(k) = Cx(k). \quad (5)$$

## Convex polyhedra

$\mathcal{V} = \{x : G_v x \leq \bar{1}\}$  (controlled-invariant),  $\mathfrak{U} = \{u : Uu \leq \bar{1}\}$ , and  $\mathfrak{D} = \{d : Dd \leq \bar{1}\}$ .

### Assumption:

- System (4) is subject to slowly-varying disturbances:  
 $\Rightarrow d(k+1) - d(k) = d_\Delta(k) \in \mathfrak{D}_\Delta = \{d_\Delta : D_\Delta d_\Delta \leq \bar{1}\}.$

### Goal

$\forall x(0) \in \mathcal{V}$ , compute  $u(k) \in \mathfrak{U}$ ,  $k = 0, 1, \dots$ , based on the available measurements such that  $x(k) \in \mathcal{V}$ ,  $\forall d(k) \in \mathfrak{D}$ ,  $\forall k \geq 0$ , and  $\lim_{k \rightarrow \infty} y(k) = r$ .

# Constant Reference Tracking: Disturbance Model

- Augmented form:

$$\underbrace{\begin{bmatrix} x(k+1) \\ d(k+1) \end{bmatrix}}_{\bar{x}(k+1)} = \underbrace{\begin{bmatrix} A & E \\ \bar{0} & I \end{bmatrix}}_{\bar{A}} \underbrace{\begin{bmatrix} x(k) \\ d(k) \end{bmatrix}}_{\bar{x}(k)} + \underbrace{\begin{bmatrix} B \\ \bar{0} \end{bmatrix}}_{\bar{B}} u(k) + \underbrace{\begin{bmatrix} \bar{0} \\ I \end{bmatrix}}_{\bar{E}} d_\Delta(k),$$

$$y(k) = \underbrace{\begin{bmatrix} C & \bar{0} \end{bmatrix}}_{\bar{C}} \begin{bmatrix} x(k) \\ d(k) \end{bmatrix}$$

or

$$\bar{x}(k+1) = \bar{A}\bar{x}(k) + \bar{B}u(k) + \bar{E}d_\Delta(k),$$

$$y(k) = \bar{C}\bar{x}(k).$$

⇒ Model commonly used in offset-free tracking Model Predictive Control (MPC);

⇒ Account for the effects of model mismatch and persistent disturbances acting on the plant;

## Assumption

The pair  $(\bar{A}, \bar{C})$  is observable and  $n_d = m$ .

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## Assumption

The pair  $(\bar{A}, \bar{C})$  is observable and  $n_d = m$ .

# Extended State Observer

Linear Observer (system state + disturbance state):

$$\bar{z}(k+1) = \bar{A}\bar{z}(k) + \bar{B}u(k) + \bar{L}[y(k) - \hat{y}(k)], \quad \hat{y}(k) = \bar{C}\bar{z}(k), \quad \bar{L} = \begin{bmatrix} L_x \\ L_d \end{bmatrix}.$$

Dynamics of the estimation error  $\bar{e}(k) = \begin{bmatrix} e_x(k) \\ e_d(k) \end{bmatrix} = \begin{bmatrix} x(k) - \hat{x}(k) \\ d(k) - \hat{d}(k) \end{bmatrix} = \bar{x}(k) - \bar{z}(k)$ :

$$\bar{e}(k+1) = (\bar{A} - \bar{L}\bar{C})\bar{e}(k) + \bar{E}d_{\Delta}(k) = \bar{A}_e\bar{e}(k) + \bar{E}d_{\Delta}(k).$$

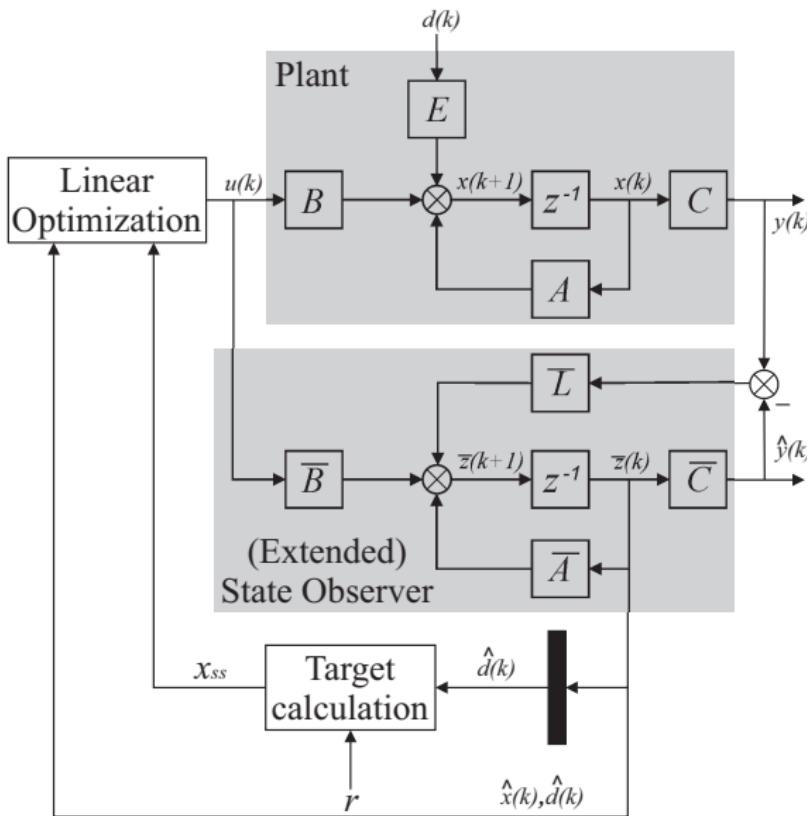
⇒ compute an mRPI set  $\bar{\mathcal{S}}_m$   $\bar{\lambda}_m$ -contractive.

⇒  $\mathcal{S}_{e_x} = \text{proj}_{e_x} \bar{\mathcal{S}}_m$ : projection of  $\bar{\mathcal{S}}_m$  onto the plant state estimation error space.

Given an OFCI pair  $(\mathcal{V}, \mathcal{S}_{e_x})$ :

- ⇒ scale  $\mathcal{S}_{e_x}$  by the largest factor  $\alpha_{e_x} > 1$  such that the pair  $(\mathcal{V}, \alpha_{e_x}\mathcal{S}_{e_x})$  remains OFCI.

# Configuration of the Tracking Control Strategy



# Tracking Target Calculation

**Assumption:** The following square matrix is nonsingular:

$$M_{ss} = \begin{bmatrix} A - I & B \\ C & \bar{0} \end{bmatrix}.$$

The pair  $(x_{ss}, u_{ss})$  is uniquely determined from the solution of the steady-state equation:

$$\begin{bmatrix} x_{ss} \\ u_{ss} \end{bmatrix} = M_{ss}^{-1} \begin{bmatrix} -E\hat{d}_{ss} \\ r \end{bmatrix}, \quad (6)$$

where  $\hat{d}_{ss}$  is the estimated disturbance in the steady-state of the observer.

# Control Action Calculation

Given an OFCI pair  $(\mathcal{V}, \alpha_{e_x} \mathcal{S}_{e_x})$ , and assuming:

- $x(k) \in \mathcal{V}$ ,
- $e(k) \in \overline{\alpha}(k) \mathcal{S}_{e_x}$ ,

compute  $u(k) \in \mathfrak{U}$  such that all admissible  $x(k+1)$  (consistent with the measurement  $y(k)$  and the bounds on the estimation error) belong to the smallest ball around the target  $x_{ss}$ :

$$\mathcal{B}(\varepsilon, x_{ss}) = \{x : -\varepsilon \leq x - x_{ss} \leq \varepsilon\} = \{x : H(x - x_{ss}) \leq \varepsilon \bar{1}\}.$$

# Control Action Calculation

worst case  $[x \ d]^T$  row by row of  $G_v$ :

$$\begin{aligned} \overline{\phi}_j(k) &= \max_{x,d} G_{v_j} [A \ E] \begin{bmatrix} x \\ d \end{bmatrix} \\ \text{s.t. } & \begin{cases} G_v x \leq \bar{1}, \ Cx = y(k), \ Dd \leq \bar{1} \\ \overline{G}_s \begin{bmatrix} x - \hat{x}(k) \\ d - \hat{d}(k) \end{bmatrix} \leq \overline{\alpha}(k) \bar{1}. \end{cases} \end{aligned}$$

worst case  $[x \ d]^T$  row by row of  $H$ :

$$\begin{aligned} \overline{\varphi}_j(k) &= \max_{x,d} H_j [A \ E] \begin{bmatrix} x \\ d \end{bmatrix} \\ \text{s.t. } & \begin{cases} G_v x \leq \bar{1}, \ Cx = y(k), \ Dd \leq \bar{1} \\ \overline{G}_s \begin{bmatrix} x - \hat{x}(k) \\ d - \hat{d}(k) \end{bmatrix} \leq \overline{\alpha}(k) \bar{1}. \end{cases} \end{aligned}$$

$$\begin{cases} G_v Bu \leq \bar{1} - \overline{\phi}(k) \quad \text{and} \quad Uu \leq \bar{1} \Rightarrow \text{for invariance} \\ HBu - \varepsilon \bar{1} \leq Hx_{ss} - \overline{\varphi}(k) \Rightarrow \text{for optimization strategy} \end{cases}$$

Control action (computed *online*)

$$\begin{aligned} u(k) &= \arg \min_{u, \varepsilon} \varepsilon \\ \text{s.t. } & \begin{bmatrix} G_v B & \bar{0} \\ U & \bar{0} \\ \hline HB & -\bar{1} \end{bmatrix} \begin{bmatrix} u \\ \varepsilon \end{bmatrix} \leq \begin{bmatrix} \bar{1} - \overline{\phi}(k) \\ \bar{1} \\ \hline Hx_{ss} - \overline{\varphi}(k) \end{bmatrix}. \end{aligned}$$

# Numerical Example

$$x(k+1) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} x(k) + \begin{bmatrix} 0 & 0.5 \\ 1 & 0.5 \end{bmatrix} u(k) + I_2 d(k), \quad y(k) = I_2 x(k)$$

- State and control constraints:

$$\Omega_x = \{x : |x_i| \leq 5, i = 1, 2.\} \quad \mathfrak{U} = \{u : |u_i| \leq 0.3, i = 1, 2.\}$$

- Bounds for disturbances and their (maximal) one step changes:

$$\mathfrak{D} = \{d : |d_i| \leq 0.1, i = 1, 2.\} \text{ and } \mathfrak{D}_\Delta = \{d_\Delta : |d_{\Delta_i}| \leq 0.01, i = 1, 2.\}$$

- $\mathcal{V} \subseteq \Omega_x$  is the maximal controlled-invariant set (**it is OFCI**), with contraction rate  $\lambda_v = 0.99$ .

- $\mathcal{S}_{e_x} = \text{proj}_{e_x} \overline{\mathcal{S}}_m$ , where:

- $\bullet$   $\overline{\mathcal{S}}_m$  is the mRPI set with contraction rate  $\overline{\lambda}_m = 0.5$ ;

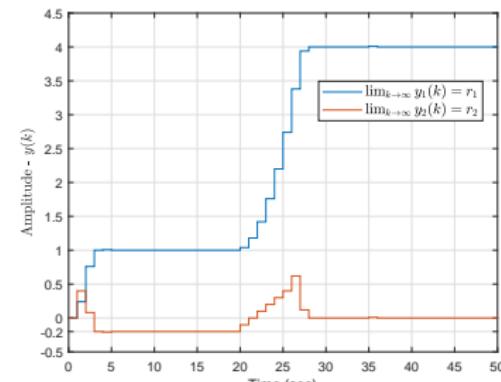
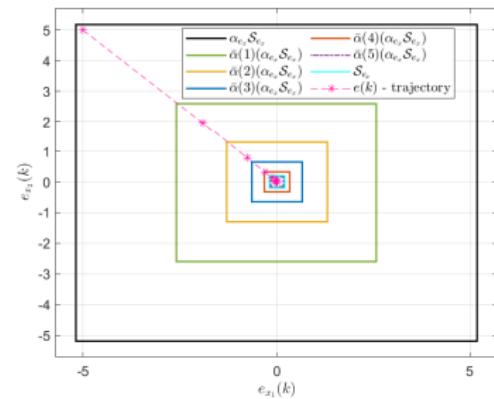
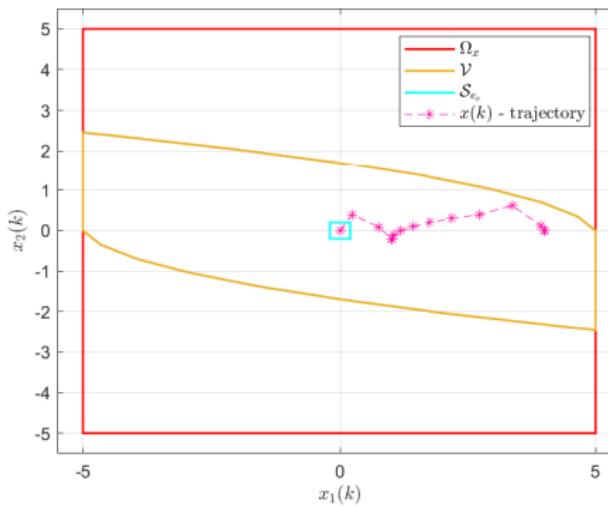
- $\bullet$   $\alpha_{e_x} = \max \alpha$  such that the pair  $(\mathcal{V}, \alpha_{e_x} \mathcal{S}_{e_x})$  is OFCI ( $\alpha_{e_x} = 26$ ).

- $e_x$  is supposed to reach and remain in the set  $\mathcal{S}_{e_x}$  after  $\bar{k}_m = 5$  steps

- $\bullet$   $\bar{x}(0) = [x(0) \mid d]^T = [0 \ 0 \mid 0.09 \ -0.05]^T$  and  $z(0) = [\hat{x}(0) \mid \hat{d}]^T = [5 \ -5 \mid 5 \ -5]^T$ .

# Numerical Example

- ⇒  $d$  is allowed to change  $d_\Delta = \pm 0.01$ ;
- ⇒ track  $r_1 = 1$  and  $r_2 = -0.2$  with  $d_1 = 0.09$  and  $d_2 = -0.05$ ;
- ⇒ after 20s,  $r_1 = 4$  and  $r_2 = 0$ ;
- ⇒ after 35s,  $d_1$  and  $d_2$  increment  $d_{\Delta_i} = 0.01$ .



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# Conclusions

- Characterization of output feedback controlled invariance of sets;
- Necessary and sufficient conditions for polyhedral OFCI;
- Observer-based dynamic controllers with larger sets of admissible initial states compared to MPC approaches;
- Design of output feedback controllers guaranteeing state constraints satisfaction;
- Constant reference tracking and disturbance rejection under constraints via output feedback.

## Points to be explored:

- Computation of an explicit output feedback control law<sup>1</sup> with reduced complexity.
- Use of OFCI sets in output feedback MPC schemes.

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<sup>1</sup>Dantas, A.D.O.S., Dantas, A.F.O.A. and Dórea, C.E.T. (2018). Static output feedback control design for constrained linear discrete-time systems using data cluster analysis. *IET Control Theory & Applications*.

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