Dwell-time stability for switched systems: from linear to (very structured) non-linear subsystems

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Outline

References

- **Section 1 (Preliminaries):**
  S. Morse (1996)
  Supervisory Control of Families of Linear Set-Point Controllers-Part 1: Exact Matching
  IEEE TAC, vol. 41, no. 10, pp. 1413-1431,
- **Section 2:**
  M. Della Rossa, L.N. Egidio and R.M. Jungers (2022)
  Stability of Switched Affine Systems: Arbitrary and Dwell-Time Switching
  arXiv preprint arXiv:2203.06968
- **Section 3:**
  M. Della Rossa and A. Tanwani (2022)
  Instability of Dwell-Time Constrained Switched Nonlinear Systems
  Systems & Control Letters (162), 105164

SECTION 1: Switched systems framework

- Given locally Lipschitz *subsystems* $f_1, f_2, \dots f_M : \mathbb{R}^n \to \mathbb{R}^n$ we consider
$$\dot{x}(t) = f_{\sigma(t)}(x(t)), \qquad \text{SwSys}$$
where $\sigma : [0, +\infty) \to \langle M \rangle := \{1, \dots, M\}$ is a switching signal.
- The switching signal "selects" which subsystems the solutions will follow. *Example:*

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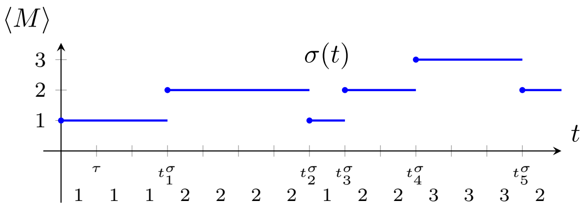

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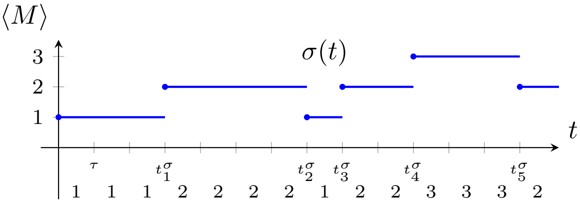

- We study uniform global asymptotic stability (UGAS) of (SwSys) *with respect to classes of switching signals*.

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- **Class of dwell time switching signals:** Given a threshold $\tau > 0$, the (so-called) *dwell-time*, consider

$$\mathcal{S}_{\mathsf{dw}}(\tau) := \left\{\sigma \in \mathcal{S} \mid t_k^\sigma - t_{k-1}^\sigma \geq \tau, \ \forall \ t_k^\sigma > 0\right\}.$$

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- Many other possible classes (periodic, average dwell times, persistence of switching conditions, etc)

The Linear Case

Consider $\mathcal{A} = \{A_1, \dots, A_M\} \subset \mathbb{R}^{n\times n}$, we consider the *linear switched system*

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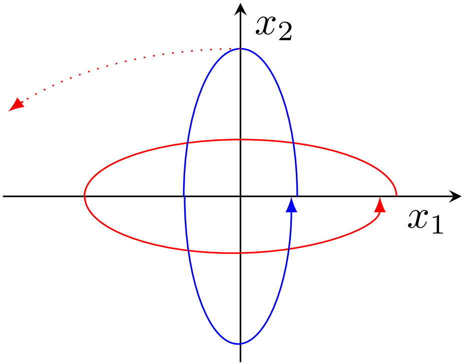
Example ([Liberzon '03, pag. 19]).
Properties/Empirical observation:

- $A_1, A_2 \in \mathbb{R}^{2\times 2}$,
- A_1, A_2 are Hurwitz stable, thus the 2 linear subsystems are exponentially stable,
- There exists a "destabilizing" $\sigma : \mathbb{R}_{\geq 0} \to \langle M \rangle$,
- If we "wait enough", we can ensure stability.



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Proposition [Morse, '96]

Given $\mathcal{A} = \{A_1, \dots, A_M\} \subset \mathbb{R}^{n \times n}$ a finite set of Hurwitz stable matrices, there exists a (large enough) $\tau_{\mathcal{A}} > 0$ such that (SwLIN) is UGAS (with respect to the origin) on $\mathcal{S}_{\mathsf{dw}}(\tau_{\mathcal{A}})$.

Remark: Lyapunov-based proof, generalizable to *exponentially stable subsystems* (not necessarily linear).

...and what about the Nonlinear Case?

Consider again the system

$$\dot{x}(t) = f_{\sigma(t)}(x(t)).$$

where $f_1, \dots, f_M : \mathbb{R}^n \to \mathbb{R}^n$ are nonlinear maps.

Question(s):

(a) What if the subsystems are *exponentially stable* with respect to different equilibria?

(b) Suppose that 0 is GAS for $\dot{x} = f_i(x)$, for all $i \in \{1, \dots, M\}$. Does it exist a (large enough) dwell time $\tau > 0$ s.t. (SwSys) is UGAS over $\mathcal{S}_{\mathsf{dw}}(\tau)$?

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“Easy” Non-linearities: For (a) we focus on affine subsystems, while for (b) we consider homogeneous subsystems.

Spoiler: Motivations/Important Features

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Section 2: Affine

- **Defn:** $f_i(x) := A_i x + b_i$
- First example of subsystems **not** sharing the same equilibrium;
- "Almost linear" when far from the origin;
- Characterize or approximate complex non-linear behaviors (via local Taylor approximation);
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Section 3: Homogeneous

- **Defn:** $f_i(\lambda x) = \lambda^k f_i(x)$, $k \in \mathbb{N}$, $\lambda \geq 0$.
- The convergence is not-exponential (can be "slower/faster");
- Positive scaling of trajectory is again a trajectory;
- Homogeneous systems provide a natural framework for the problem of finite-time (or practical) stabilization.

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- $x_{ei} = -A_i^{-1}b_i$ is the equilibrium of the i-th subsystem;
- The asymptotic behavior of (SwAFF) is related with the one of

$$\dot{x}(t) = A_{\sigma(t)}x(t), \qquad \text{SwLIN}$$

called the *linearization of system* (SwAFF).

Arbitrary Switching Stability

Main Idea: Under the assumption that (SwLIN) is UGAS, we show that (SwAFF) has an exponentially stable compact set.

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- **Notation:** $\Psi_\sigma(t,x)$ ($\Psi_\sigma^{lin}(t,x)$) is the sol. of (SwAFF) (resp. (SwLIN)).
- A compact set C is *uniformly exponentially stable* for (SwAFF) on $\widetilde{\mathcal{S}}$ if there exists a $M>0$ and $\kappa>0$, for all $x\in\mathbb{R}^n$, all $\sigma\in\widetilde{\mathcal{S}}$ and all $t\in\mathbb{R}_{\geq 0}$, it holds that

$$|\Psi_\sigma(t,x)|_C \leq M\,|x|_C\,e^{-\kappa t}.$$

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Technical Lyapunov result for *Linear* Sw. Sys., [Mol. and Pya., '89]

(SwLIN) is UGAS on $\mathcal{S}$ if and only if there exists a *norm* $v : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ and a scalar $\kappa > 0$ such that

$$v(\Psi_\sigma^{lin}(t,x)) \leq e^{-\kappa t} v(x),\ \forall x \in \mathbb{R}^n,\ \forall t \in \mathbb{R}_{\geq 0},\ \forall \sigma \in \mathcal{S}.$$

Existence Result

The proof basically follows the idea of the discrete time case.

First Lemma->a "safety" outer bound

Suppose that linearized system (SwLIN) is UGAS on $\mathcal{S}$. Take $v : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ and $\kappa > 0$ as in the Converse Lyapunov Lemma. Then there exists $R > 0$ such that $\mathcal{K}_{v,R} := \{x \in \mathbb{R}^n \mid v(x) \leq R\}$, is forward invariant for (SwAFF) on $\mathcal{S}$.

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With this security region, we are able to provide the existence result.

Theorem: Existence of exponentially stable set under arbitrary switching

Consider the set $\mathcal{K}(t) = \{\Psi_\sigma(t, 0) \mid \sigma \in \mathcal{S}\}$. (reachable set from 0 at time t). Then

$$\mathcal{K}_\infty = \lim_{t\to\infty} \mathcal{K}(t)$$

is well-defined, and it is uniformly exponentially stable.

Some properties of $\mathcal{K}_\infty$ and example

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- $\mathcal{K}_\infty$ contains all the equilibria of all the convex combinations of subsystems (a.k.a. the Filippov equilibria) (as expected.)
- $\mathcal{K}_\infty$ is compact, (path-)connected, but in general *not* convex!
- Our proof (based on a set-limit) is not "numerically constructive". We propose two different methods, providing (possibly non-convex) outer approximations.

Two methods for over-approximating $\mathcal{K}_\infty$

LMI-based

SOS-based

Proposition 1

If there exists a symmetric matrix $S \in \mathbb{R}^{n\times n}$ a vector $c \in \mathbb{R}^n$ and a scalar $\kappa > 0$ satisfying

$$SA_i^\top + A_iS \prec -2\kappa S, \qquad \forall i \in \langle M \rangle$$

$$\begin{bmatrix} \kappa^2 & (A_ic + b_i)^\top \\ A_ic + b_i & S \end{bmatrix} \succ 0, \quad \forall i \in \langle M \rangle$$

then $\mathcal{K}_Q := \{x \in \mathbb{R}^n \ : \ (x-c)^\top S^{-1}(x-c) \leq 1\} \supseteq \mathcal{K}_\infty$ is forward invariant.

Proposition 2

If there exist a polynomial $V(x) \in \mathbb{R}[x]$ of degree d and $r > 0$, $\beta \geq 0$, $\varepsilon > 0$ satisfying

$$V(x) - \epsilon\|x\|_d^d \text{ is SOS}$$

$$-\nabla V^\top(x)(A_ix + b_i) - \beta(V(x) - r) \text{ is SOS}$$

$\forall i \in \langle M \rangle$ then

$$\mathcal{K}_{\mathrm{SOS}} := \{x \in \mathbb{R}^n \ : \ V(x) \leq r\} \supseteq \mathcal{K}_\infty$$

is forward invariant.

Example, Outer Estimation of $\mathcal{K}_\infty$

We studied the planar 2-modes example given by $A_1 = \begin{bmatrix} -1 & -1 \\ 0 & -1 \end{bmatrix}$, $A_2 = \begin{bmatrix} -1 & 0 \\ -1 & -1 \end{bmatrix}$, $b_1 = b_2 = [-1, -1]^\top$. The two equilibria are $x_{e1} = [0, -1]^\top$ and $x_{e2} = [-1, 0]^\top$.

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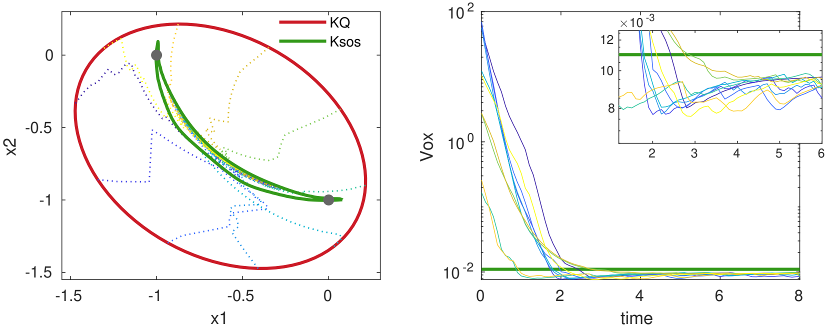

Figure: On the left, a representation of state space with the forward invariant sets K_Q and K_{SOS}. On the right, the evaluation of polynomial $V(x)$ associated to K_{SOS} along trajectories.

DWELL-TIME: Preliminaries

Recall: Given a time threshold $\tau > 0$, a so-called *dwell-time*, consider

$$\mathcal{S}_{\mathsf{dw}}(\tau) := \left\{\sigma \in \mathcal{S} \mid t_k^\sigma - t_{k-1}^\sigma \geq \tau,\ \forall\, t_k^\sigma > 0\right\},$$

where $\{t_k^\sigma\}$ denotes the set of time instants at which σ is discontinuous.

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Technical Lyapunov result for *Linear* Sw. Sys. [Wirth, 2005]

Given $\tau > 0$, (SwLIN) is UGAS on $\mathcal{S}_{dw}(\tau)$ if and only if there exist $\kappa > 0$ and norms $v_1 \dots v_M : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ such that

$$\begin{aligned} v_i(e^{A_i t}x) &\leq e^{-\kappa t} v_i(x), \quad \forall x \in \mathbb{R}^n,\ \forall\, t \in \mathbb{R}_{\geq 0},\ \forall i \in \langle M \rangle. \\ v_i(e^{A_i \tau}x) &\leq e^{-\kappa \tau} v_j(x), \quad \forall x \in \mathbb{R}^n,\ \forall (i,j) \in \langle M \rangle^2. \end{aligned}$$

Stability/Asymptotic Analysis

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(Weaker) Stability/ Boundedness Notion

Given a class $\mathcal{S}_{\mathsf{dw}}(\tau)$, (SwAFF) is *uniformly globally ultimately bounded (UGUB) on* $\mathcal{S}_{dw}(\tau)$ if there exists a compact set $\mathcal{V} \subset \mathbb{R}^n$ such that

$$\forall x \in \mathbb{R}^n,\ \forall \sigma \in \mathcal{S}_{\mathsf{dw}}(\tau),\ \exists\, T(\sigma, x) \geq 0 \text{ such that } \forall t \geq T(\sigma, x),\ \ \Psi_\sigma(t, x) \in \mathcal{V}.$$

In this case the compact set $\mathcal{V} \subset \mathbb{R}^n$ is said to be a *uniform bounding region*.

- $\mathcal{V}$ is not necessary forward invariant (and in our case it is not, in general), but all solutions, at a certain instant (depending on the particular solution), enter and stay inside it.

Existence of Bounding Regions

Existence Theorem

For any given $\tau \in \mathbb{R}_{\geq 0}$, suppose that the linearized system (SwLIN) is UGAS on $\mathcal{S}_{\mathsf{dw}}(\tau)$. Then (SwAFF) is uniformly globally ultimately bounded (UGUB) on $\mathcal{S}_{\mathsf{dw}}(\tau)$.

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The proof is skipped, it uses of the previous Lyapunov conv. result and the Lemma:

Technical Lemma

For $\tau > 0$, suppose that (SwLIN) is UGAS on $\mathcal{S}_{\mathsf{dw}}(\tau)$. Then, there exist translated norms[a] $\widetilde{v}_i : \mathbb{R}^n \to \mathbb{R}$, a $\widetilde{\kappa} > 0$ and compact sets $\mathcal{X}_i \subset \mathbb{R}^n$, $i \in \langle M \rangle$, such that

$$
\begin{gathered}
x_{ei} \in \mathrm{Int}(\mathcal{X}_i), \ \ \forall\, i \in \langle M \rangle, \\
\widetilde{v}_i\big(\Psi_i(t,x)\big) \leq \widetilde{v}_i(x), \quad \forall x \in \mathbb{R}^n \setminus \mathrm{Int}(\mathcal{X}_i), \ \forall t \in \mathbb{R}_{\geq 0}, \forall i \in \langle M \rangle, \\
\widetilde{v}_i\big(\Psi_i(\tau,x)\big) \leq e^{-\widetilde{\kappa}\tau}\widetilde{v}_j(x), \quad \forall x \in \mathbb{R}^n \setminus \mathrm{Int}(\mathcal{X}_j), \ \forall (i,j) \in \langle M \rangle^2.
\end{gathered}
$$

[a] A function $w : \mathbb{R}^n \to \mathbb{R}$ is said to be a *translated norm* if there exist a norm $v : \mathbb{R}^n \to \mathbb{R}$ and a vector $c \in \mathbb{R}^n$ (called the *center* of w) such that $w(x) = v(x - c)$, for all $x \in \mathbb{R}^n$.

Graphical Representation

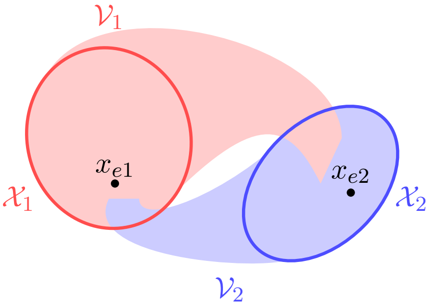

Figure: For two subsystems $\{1, 2\}$, representation of the sets $\mathcal{X} = \mathcal{X}_1 \cup \mathcal{X}_2$ (solid lines), level subsets of the translated norms. The bounding region is defined as the union $\mathcal{V} = \cup_{i \in \langle M \rangle} \mathcal{V}_i$

Forward Invariance with respect to the switching instants

We can also show that the region $\mathcal{X} := \cup_{i\in\langle M\rangle}\mathcal{X}_i$ has the following property:

A "weaker" forward invariance property

A compact set $C \subset \mathbb{R}^n$ is *forward invariant* for (SwAFF) on $\mathcal{S}_{\mathsf{dw}}(\tau)$ *with respect to the switching points* if, for all $x \in C$, all $\sigma \in \mathcal{S}_{\mathsf{dw}}(\tau)$, we have that

$$\Psi_\sigma(t_k^\sigma, x) \in C, \ \ \forall x \in C, \ \ \forall \sigma \in \mathcal{S}_{\mathsf{dw}}(\tau), \ \ \forall t_k^\sigma \geq 0,$$

where, we recall, $\{t_k^\sigma\}$ denotes the (finite or countable) set of discontinuities of the signal $\sigma \in \mathcal{S}_{\mathsf{dw}}(\tau)$.

Estimation of bounding regions via LMIs.

Inspired by LMI sufficent (but unfortunately, not necessary, conditions) proposed in [Geromel & Colaneri '06'], we restrict the search on quadratic (translated) norms.

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LMIs conditions

Look for positive definite matrices $P_i, W_{ij} \in \mathbb{R}^{n \times n}$ and vectors $c_i, d_{ij} \in \mathbb{R}^n$ satisfying the inequalities

$$\begin{aligned} \mathcal{A}_i' \mathcal{P}_i + \mathcal{P}_i \mathcal{A}_i &\prec -\mathcal{E}_{ii} && \forall i \in \langle M \rangle \\ e^{\mathcal{A}_i' \tau} \mathcal{P}_i e^{\mathcal{A}_i \tau} - \mathcal{P}_j &\prec -\mathcal{E}_{ij} && \forall (i,j) \in \langle M \rangle^2, i \neq j \end{aligned}$$

with

$$\mathcal{P}_i = \begin{bmatrix} P_i & -P_i c_i \\ -c_i^\top P_i & c_i^\top P_i c_i \end{bmatrix}, \; \mathcal{A}_i = \begin{bmatrix} A_i & b_i \\ 0 & 0 \end{bmatrix}, \; \mathcal{E}_{ij} = \begin{bmatrix} W_{ij} & -W_{ij} d_{ij} \\ -d_{ij}^\top W_{ij} & d_{ij}^\top W_{ij} d_{ij} - 1 \end{bmatrix}.$$

Examples and Discussion

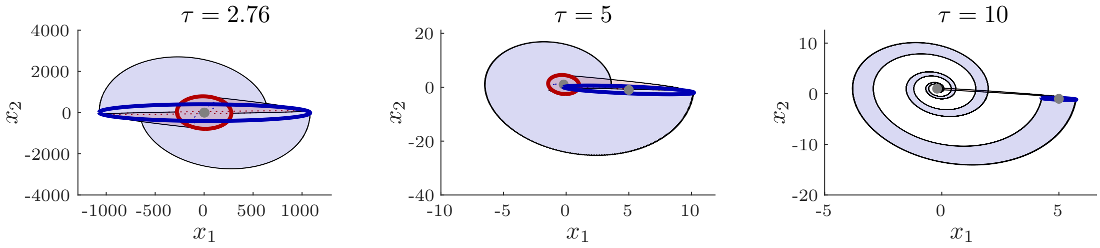
We consider a switched affine system (SwAFF) defined by

$$A_1 = \begin{bmatrix} 0 & 1 \\ -10 & -1 \end{bmatrix},\; A_2 = \begin{bmatrix} 0 & 1 \\ -0.1 & -0.5 \end{bmatrix},\; b_1 = \begin{bmatrix} -1 \\ -1 \end{bmatrix},\; b_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Examples and Discussion

We consider a switched affine system (SwAFF) defined by

$$A_1 = \begin{bmatrix} 0 & 1 \\ -10 & -1 \end{bmatrix},\ A_2 = \begin{bmatrix} 0 & 1 \\ -0.1 & -0.5 \end{bmatrix},\ b_1 = \begin{bmatrix} -1 \\ -1 \end{bmatrix},\ b_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$



For three different values of τ, the regions X_1 (red line), X_2 (blue line), $\mathcal{V}_1$ (red area) and $\mathcal{V}_2$ (blue area) are represented. The bounding region $\mathcal{V} = \mathcal{V}_1 \cup \mathcal{V}_2$ is the UGUB set.

3-Mode example: Optimizing centers of translated norms.

$$A_1 = \begin{bmatrix} -5 & 1 \\ -1 & -4 \end{bmatrix},\; A_2 = \begin{bmatrix} -5 & -1 \\ 1 & -4 \end{bmatrix},\; A_3 = \begin{bmatrix} -2 & 8 \\ -5 & -5 \end{bmatrix},\; b_1 = \begin{bmatrix} -50 \\ -10 \end{bmatrix},\; b_2 = \begin{bmatrix} -10 \\ -40 \end{bmatrix},\; b_3 = 0.$$

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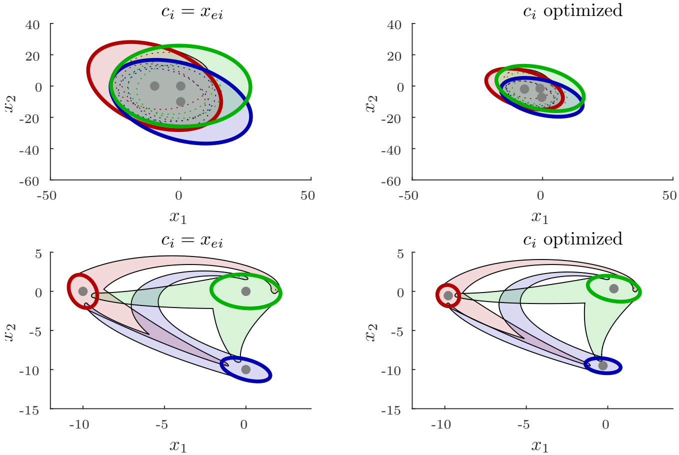
- Regions $\mathcal{X}_1$, $\mathcal{X}_2$ and $\mathcal{X}_3$ and regions $\mathcal{V}_1$, $\mathcal{V}_2$ and $\mathcal{V}_3$ for $\tau = 0.1$ (top) and $\tau = 0.5$ (bottom).
- On the left $c_i = x_{ei}$ for all $i \in \langle M \rangle$, on the right c_i "optimized".



Summary of Section 2

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- Proof Technique: Use the strong properties of linear switching systems to say something about affine ones.
- Semidefinite optimization approaches to provide “safe” outer approximations.

SECTION 3: Homogeneous Case

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We study the switched system

$$\dot{x}(t) = f_{\sigma(t)}(x(t)), \qquad \text{SwHom}$$

k-Homogeneity Assumption

Given $k \in \mathbb{R}$, $f_i : \mathbb{R}^n \to \mathbb{R}^n$, is *homogeneous of degree* k (and we write $f_i \in \mathcal{H}_n^k$), i.e.

$$f_i(\lambda x) = \lambda^k f_i(x), \quad \forall x \in \mathbb{R}^n, \; \forall \lambda > 0.$$

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- $k = 1$ corresponds to the "linear grown" case. Earlier results (Morse '96) apply.
- $k > 1$ corresponds to superlinear case. Slow decay near the origin.
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QUESTION: Suppose that $\dot{x} = f_i(x)$ is GAS, for all $i \in \langle M \rangle$. Does it exist a (large enough) dwell time $\tau > 0$ s.t. (SwHom) is UGAS over $\mathcal{S}_{\mathrm{dw}}(\tau)$?

Motivating/Illustrating Example

For $i \in \{1,2\}$, consider

$$f_i^k(x) := |A_i x|^{k-1} A_i x$$

where $A_1, A_2 \in \mathbb{R}^{2\times 2}$ are Hurwitz matrices, chosen as in [Liberzon '03, pag. 19].

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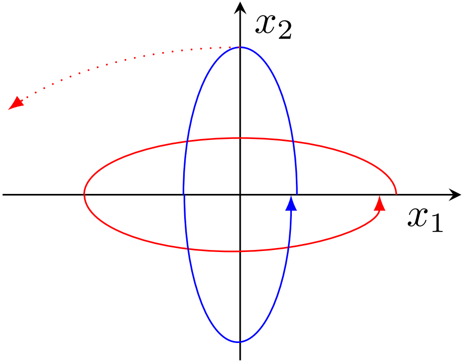
Properties:

- $k = 1$ corresponds to the example already presented!;
- It is NOT arbitrary stable; (UGAS on $\mathcal{S}$);
- There exists a (large enough) dwell-time $\tau > 0$ such that it is UGAS on $\mathcal{S}_{\text{dw}}(\tau)$



$k > 1$: Instability but Ultimate Boundedness

For $i \in \{1, 2\}$, consider

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Key Observation: Same trajectories, but "fast" far from the origin, "slow" close to it.

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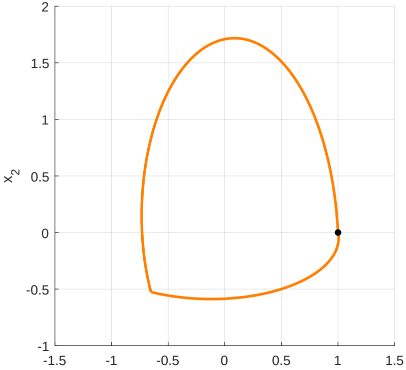
Idea:

- There exists a periodic non-converging solution, corresponding to a periodic (and, in particular, dwell-time) signal;
- By scaling the initial condition, the period/dwell-time can be increased arbitrarily.
- This time-scaling does **not** occur in linear vector fields.



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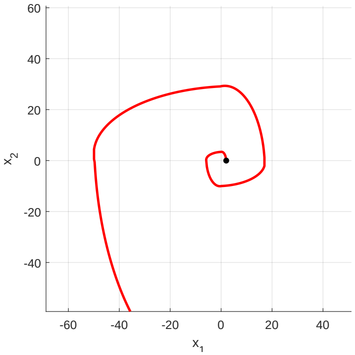
Idea:

- We can build, as in the linear case, a destabilizing signal.
- By scaling, the period/dwell-time can be increased arbitrarily.
- **Formally:** For any $\tau > 0$, there exist $x_0 \in \mathbb{R}^n$ and $\sigma \in \mathcal{S}_{\mathsf{dw}}(\tau)$ such that the corresponding solution diverges.



Main Theorem, “Bad” behavior for any dwell-time

Qualitative Behavior, SuperLinear case

Given $\mathcal{F} = \{f_i\}_{i\in\langle M\rangle} \subset \mathcal{H}_n^k$, with $k > 1$, and suppose that $\dot{x} = f_i(x)$ is GAS, for each $i \in \langle M\rangle$. Then, the following hold:

- (*Ultimate boundedness*) For every $\tau > 0$ there exists an $R(\tau) > 0$ such that, for each $x_0 \in \mathbb{R}^n$ and $\sigma \in \mathcal{S}_{\mathsf{dw}}(\tau)$
$$\limsup_{t\to+\infty} |\phi_{\mathcal{F}}(t, x_0, \sigma)| \leq R(\tau).$$
- (*Non-Stability*) Generally, for every $\tau > 0$, there exists a ball $\mathbb{B}(0, \overline{R}(\tau))$, a sequence $(x_{0\ell})_{\ell\in\mathbb{N}}$, $x_{0\ell} \to 0$, $\sigma_\ell \in \mathcal{S}_{\mathsf{dw}}(\tau)$ and $t_\ell > 0$ such that $\phi_{\mathcal{F}}(t_\ell, x_{0\ell}, \sigma_\ell) \notin \mathbb{B}(0, \overline{R}(\tau))$ (i.e. 0 is *not* Lyapunov stable).

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These (“good/bad”) qualitative properties hold *for any* dwell-time. Of course, the “safety” and “instability” radii $R(\tau)$, $\overline{R}(\tau)$ depend on the chosen dwell time.

“Dual” Result, the Sub-Linear Case

Sub-Linear Case

Consider $\mathcal{F} = \{f_i\}_{i\in\langle M\rangle} \subset \mathcal{H}_n^k$, with $k < 1$, $k \neq 0$ and suppose that the subsystem $\dot{x} = f_i(x)$ is GAS, for each $i \in \langle M\rangle$. Then:

- (*Local Asymptotic Stability*) For every $\tau > 0$, there exists an $r(\tau) > 0$ such that the origin is a uniform (local) asymptotically stable equilibrium in $\mathbb{B}(0, r(\tau))$ of (SwHom) on $\mathcal{S}_{\mathsf{dw}}(\tau)$;
- (*Diverging Solutions*) In general, for every $\tau > 0$, there exists $z_0 \in \mathbb{R}^n$ and $\sigma \in \mathcal{S}_{\mathsf{dw}}(\tau)$ such that $\limsup_{t\to\infty} |\phi_{\mathcal{F}}(t, z_0, \sigma)| = +\infty$.

Again, the radius $r(\tau)$ and the norm of the “problematic” initial conditions, will depend on the dwell time τ.

Proof Technique: “Translation of Solutions”

Reduction to degree 1

Consider $f \in \mathcal{H}_n^k$, $(k \neq 0)$ define the *reduction of degree 1 of f* as the function $g_f \in \mathcal{H}_n^1$ defined by $g_f(0) = 0$ and $g_f(x) := \frac{|f(x)|^{\frac{1}{k}}}{|f(x)|} f(x), \;\; \forall\, x \in \mathbb{R}^n \setminus \{0\}$.

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Translation of solutions

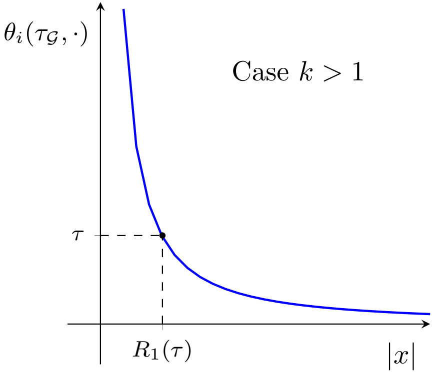
Given f, its reduction g_f, there exists a time-scaling map $\theta : \mathbb{R}_{\geq 0} \times (\mathbb{R}^n \setminus \{0\}) \to \mathbb{R}$ s.t.

$$\phi_g(t,x) = \phi_f(\theta(t,x), x), \quad \forall x \neq 0, \forall t \in \mathbb{R}_{\geq 0}.$$

- Given a homogeneous switched system we consider its reduction of degree 1, for which classical results ([Morse '96]) hold;
- Using the function $\theta : \mathbb{R}_{\geq 0} \times (\mathbb{R}^n \setminus \{0\}) \to \mathbb{R}$ we have qualitative properties of the original system.
- The properties of θ strongly depend on $(k > 1$ or $k < 1)$.

Qualitative Behavior of θ

We use θ to construct safety/local stability and unboundedness/instability radii.



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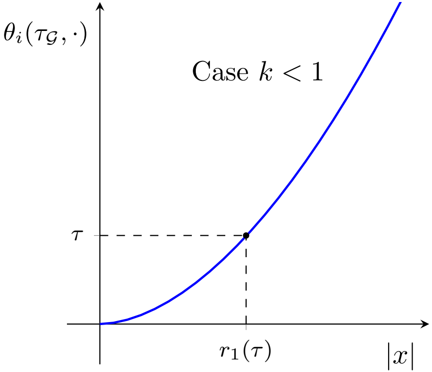

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Semi-global & practical dwell-time stability

Consider $\mathcal{F} = \{f_i\}_{i\in\langle M\rangle} \subset \mathcal{C}^1(\mathbb{R}^n \setminus \{0\}, \mathbb{R}^n)$ and suppose that $\dot x = f_i(x)$ is GAS, for each $i \in \langle M\rangle$. For every $M > \varepsilon > 0$ there exists a $\tau = \tau(\varepsilon, M) > 0$ such that

$$\limsup_{t\to+\infty} |\phi_{\mathcal{F}}(t, x_0, \sigma)| \leq \varepsilon, \ \forall |x_0| \leq M, \ \forall \sigma \in \mathcal{S}_{\mathsf{dw}}(\tau). \qquad \spadesuit$$

Similarly, for every $\tau > 0$ there exist $\varepsilon = \varepsilon(\tau) > 0$ and $M = M(\tau) > 0$ for which ($\spadesuit$) holds.

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- Adapting the proof techniques/ideas for more general classes of non-linear systems.
- Affine switching systems, or more in general, subsystems *not* sharing the same equilibrium as model for a multi-target game.

Thank you!

And thanks to: