Results and thoughts concerning stabilization of discrete-time linear switched systems

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joint work with

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May 11th 2022

This talk is based on:

- R. Jüngers, P.M., "On feedback stabilization of linear switched systems via switching signal control", SIAM SICON 2017;
- C. Dettmann, R. Jüngers, P.M., "Lower bounds and dense discontinuity phenomena for the stabilizability radius of linear switched systems.", SCL 2020;
- P.M., "Some remarks on the stabilisation problem and the stabilisability radius for linear switched systems", ICSC 2022.

Context: discrete-time linear switched systems

Linear switched systems:

$$x_{k+1} = A_{\sigma(k)} x_k, \quad x_k \in \mathbb{R}^n$$

- the map $\sigma : \mathbb{N} \to \{1, \dots, m\}$ is the *switching signal*
- A_σ takes values on a set of matrices $\mathcal{M} = \{A_1, \dots, A_m\} \subset M_n(\mathbb{R})$. A_i are also called *modes*

Typical problems concern stability/stabilization of switched systems. Let

$$\rho(A) = \max\{|\lambda_1(A)|, \dots, |\lambda_n(A)|\} \leftarrow \text{ spectral radius of } A$$

It is well-known that switching among stable matrices, i.e. $\rho(A_i) < 1, \ \forall i$, may produce an unstable behavior.

Similarly, it may be possible to stabilize the system from a given initial state by switching among unstable modes.

"Measures" of the stability properties

$\rho(A) \to$ "worst" exponential rate of the linear dynamics $x_{k+1} = Ax_k$

For a linear switched system the worst exponential rate is given by the **joint spectral radius** (Rota & Strang, 1960)

$$\rho(\mathcal{M}) = \lim_{k\to\infty} \sup_{M\in\mathcal{M}^k} \|M\|^{1/k} = \limsup_{k\to\infty} \sup_{M\in\mathcal{M}^k} \rho(M)^{1/k}$$

where $\mathcal{M}^k =$ products of k matrices in $\mathcal{M}$. The joint spectral radius has applications in several fields (e.g. it is related to the class of regularity of wavelets)

Computation of $\rho(\mathcal{M})$ is NP-hard and the question $\rho(\mathcal{M}) \leq 1$ is undecidable (no algorithm to determine it). However in most of the cases known algorithms allow to compute it efficiently (even exactly)

Measures of stabilizability

A first measure of the stabilizability of the system is given by the **joint spectral subradius** (Gurvits, 1995):

$$\check{\rho}(\mathcal{M}) = \lim_{k\to\infty} \inf_{M\in\mathcal{M}^k} \|M\|^{1/k} = \lim_{k\to\infty} \inf_{M\in\mathcal{M}^k} \rho(M)^{1/k}$$

The joint spectral subradius measures the smallest growth of the system <u>without reference to x_0.</u>

Similarly to the joint spectral radius, there exist formal negative results about the computation of the joint spectral subradius.

Stabilizability radius

Joint spectral subradius measures the smallest growth of the system without reference to x_0

Assume that we may chose the switching signal depending on $x_0 \in \mathbb{R}^n$:

$$\tilde{\rho}_{x_0}(\mathcal{M}) = \inf \left\{ \lambda > 0 \,\middle|\, \begin{array}{c} \exists C > 0, \exists x_{(\cdot)} \text{ traj. of the switched sys.} \\ s.t.\ |x_k| \leq C\lambda^k |x_0| \quad \forall k \geq 0 \end{array} \right\}$$

stabilizability radius
(Jungers - M. 2017) $\rightarrow \tilde{\rho}(\mathcal{M}) = \sup_{x_0} \tilde{\rho}_{x_0}(\mathcal{M})$

Properties of the stabilizability radius *

It can be proved that the constant C in the def. of $\tilde{\rho}_{x_0}(\mathcal{M})$ can be assumed independent of x_0:

$$\tilde{\rho}(\mathcal{M}) = \inf \left\{ \lambda > 0 \,\middle|\, \begin{array}{c} \exists C > 0,\ s.t.\ |x_k| \leq C\lambda^k |x_0| \quad \forall x_0, \forall k \geq 0 \\ \text{for some switching law depending on } x_0 \end{array} \right\}$$

Furthermore

$$\begin{aligned} \tilde{\rho}(\mathcal{M}) < 1 \quad &\iff \quad \forall x_0 \text{ there exists a switching law s.t } x_k \to 0 \\ &\iff \quad \exists \sigma(x) \text{ s.t. } x_k \to 0 \text{ where } x_{k+1} = A_{\sigma(x_k)} x_k \\ &\iff \quad \exists V \text{ control-Lyapunov function} \end{aligned}$$

Other properties:

- $\tilde{\rho}(\gamma \mathcal{M}) = \gamma \tilde{\rho}(\mathcal{M})$ and $\tilde{\rho}(\mathcal{M}^k) = \tilde{\rho}(\mathcal{M})^k$.
- the problem $\tilde{\rho}(\mathcal{M}) < 1$ is undecidable

* Jungers & M. 2017. See also Sun & Ge 2011 for similar results

Exemple (based on Stanford & Urbano, 1994)

Define $\mathcal{M} = \{A_1, A_2\}$ as

$$A_1 = \begin{pmatrix} \cos\frac{\pi}{4} & \sin\frac{\pi}{4} \\ -\sin\frac{\pi}{4} & \cos\frac{\pi}{4} \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$$

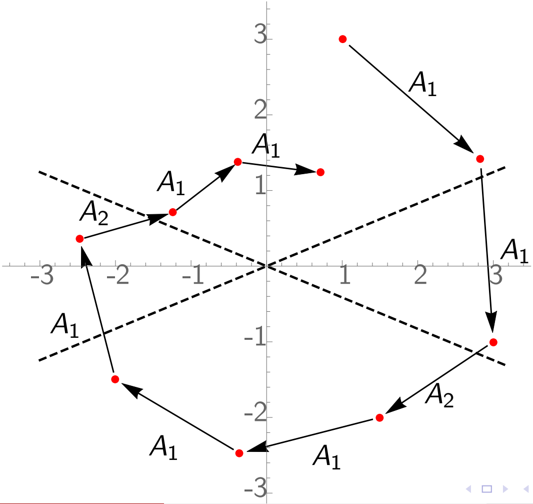
$$A_2 = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 2 \end{pmatrix}$$

Both A_1, A_2 preserve the area and it is easy to see that $\check{\rho}(\mathcal{M}) = 1$.

To stabilize the system from a point x_0 rotate with A_1 at most 3 times until the angle with the x-axis $\leq \frac{\pi}{8}$ and then apply A_2 $\Rightarrow$ the induced norm contraction gives the bound $\tilde{\rho}(\mathcal{M}) \leq (\frac{34-15\sqrt{2}}{16})^{1/8} < 0.9724$

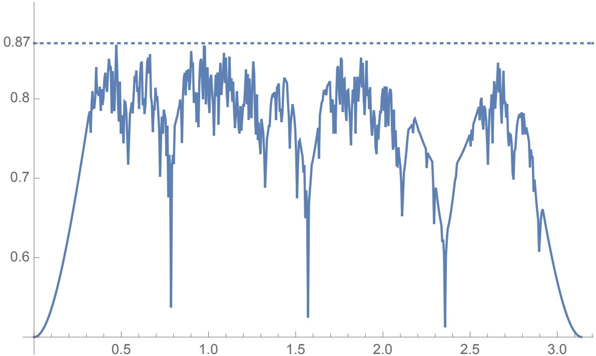
Exemple (Stanford & Urbano, 1994)

Stabilizing feedback with $A_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$, $A_2 = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 2 \end{pmatrix}$



Exemple (Stanford & Urbano, 1994)

Given an integer $k > 1$ the value $\max_{x_0 \in \mathbb{S}^1} \min(\frac{|x_k|}{|x_0|})^{\frac{1}{k}}$ computed with the previous feedback is an upper bound for $\tilde{\rho}(\mathcal{M})$. For $k = 30$ this gives $\tilde{\rho}(\mathcal{M}) \leq 0.87$.



Numerical approximation of $\tilde{\rho}(\mathcal{M})$

$\tilde{\rho}(\gamma\mathcal{M}) = \gamma\tilde{\rho}(\mathcal{M}) \Rightarrow$ in order to approximate $\tilde{\rho}(\mathcal{M})$ one may use algorithms to check stabilizability, e.g.:

- Lyapunov-Metzler inequalities (Geromel & Colaneri 2006)

- Existence of piecewise quadratic control-Lyapunov functions and asymptotically tight algorithm for stabilizability
(Zhang, Abate, Hu & Vitus, 2009)

- "Invariance" conditions and related results/stabilization algorithms (Fiacchini & Jungers 2014, Fiacchini, Girard & Jungers 2016):

$$\exists \Omega \subset \mathbb{R}^n \ s.t. \ \Omega \subset \mathrm{int}(\cup_{M \in \cup_{i \leq k} \mathcal{M}^i} M^{-1}\Omega)$$

The previous methods provide upper bounds for $\tilde{\rho}(\mathcal{M})$, but the rate of convergence to the true value is unclear

A simple lower bound for $\tilde{\rho}(\mathcal{M})$

Let $0 \leq s_1(A) \leq s_2(A) \leq \cdots \leq s_n(A) = \|A\|$ denote the singular values of $A \in M_n(\mathbb{R})$, that is

$$s_i(A) = \sqrt{\lambda_i(A^T A)}$$

($s_i(A)$ = lengths of semiaxes of the ellipsoid $A \cdot B_1(0)$ with $B_1(0)$ unit ball)

Theorem 1 (Jungers & M. 2017)

$$\tilde{\rho}(\mathcal{M}) \geq \tilde{\rho}_1(\mathcal{M}) \triangleq \min_{i=1,\ldots,m} s_1(A_i)$$

Example:

$$\tilde{\rho}\left(\left\{\frac{\sqrt{2}}{2}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}, \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 2 \end{pmatrix}\right\}\right) \geq \frac{1}{2}$$

For this example using $\tilde{\rho}(\mathcal{M}) = \tilde{\rho}(\mathcal{M}^k)^{1/k} \geq (\min_{A \in \mathcal{M}^k} s_1(A))^{1/k} = \frac{1}{2}$ does not improve the bound

Second lower bound for $\tilde{\rho}(\mathcal{M})$

From now on we assume that the matrices A_i are **invertible**

Theorem 2 (Dettmann, Jungers & M. 2020)

The stabilizability radius satisfies

$$\tilde{\rho}(\mathcal{M}) \geq \tilde{\rho}_2(\mathcal{M}) \triangleq \left(\sum_{k=1}^{m} |\det A_k|^{-1}\right)^{-1/n}$$

Example:
$$\tilde{\rho}\left(\left\{\frac{\sqrt{2}}{2}\begin{pmatrix}1 & 1\\ -1 & 1\end{pmatrix}, \begin{pmatrix}\frac{1}{2} & 0\\ 0 & 2\end{pmatrix}\right\}\right) \geq 2^{-1/2} \approx 0.707$$

$\rightarrow$ provide significant improvement compared to Theorem 1

Thm 2 not always better than Thm 1: if $\mathcal{M} = \{\text{two rotation matrices}\}$ then Thm 1 returns a lower bound 1, while Thm 2 returns $\sqrt{2}/2$

Sketch of the proof

Given $A \in M_n(\mathbb{R})$ define $\mathbb{S}_{r,A} = \{x \in \mathbb{S}^{n-1} \mid \|Ax\| \leq r\}$. It is empty if $r < s_1(A)$. Otherwise one can show the following bound of its measure

$$m(\mathbb{S}_{r,A}) \leq m(\mathbb{S}^{n-1}) \min\{r^n |\det(A)|^{-1}, 1\} \tag{1}$$

If $\rho > \tilde{\rho}(\mathcal{M})$ and we set $r = \rho^k$ then $\mathbb{S}^{n-1}$ must be covered by the union of such sets for $A \in \mathcal{M}^k$ for $k \to \infty$, so that

$$m(\mathbb{S}^{n-1}) \leq \sum_{A \in \mathcal{M}^k} m(\mathbb{S}_{\rho^k, A}). \tag{2}$$

Determinants are multiplicative $\Rightarrow$

$$\sum_{A \in \mathcal{M}^k} |\det(A)|^{-1} = \sum_{\sigma \in \{1,\dots,m\}^k} \prod_{i=1,\dots,k} |\det(A_{\sigma_i})|^{-1} = \Big(\sum_{h=1}^{m} |\det(A_h)|^{-1} \Big)^k \tag{3}$$

From (1), (2) and (3) one gets the stated bound. $\square$

Improved lower bound

Let
$$\Sigma_m = \{\nu \in [0,1]^m \mid \sum_{i=1}^m \nu_i = 1\}, \quad \bar\nu \in \Sigma_m \;\; s.t. \;\; \bar\nu_h = \frac{|\det A_h|^{-1}}{\sum_{i=1}^m |\det A_i|^{-1}}$$
$$\Psi(\nu) = \sum_{i=1}^m \nu_i \log\Big(\frac{\nu_i|\det A_i|}{s_1(A_i)^n}\Big), \quad \mathcal{Z} = \{\nu \in \Sigma_m \mid \Psi(\nu) = 0\}$$

Set $\tilde\rho_3(\mathcal{M}) \triangleq \min_{\nu\in\mathcal{Z}} \prod_{i=1}^m s_1(A_i)^{\nu_i}$. Also, recall $\tilde\rho_1(\mathcal{M}) \triangleq \min_{i=1,\dots,m} s_1(A_i)$ from Thm 1 and $\tilde\rho_2(\mathcal{M}) \triangleq \big(\sum_{i=1}^m |\det A_i|^{-1}\big)^{-1/n}$ from Thm 2.

Theorem 3 (Dettmann, Jungers & M. 2020)

We have the following alternative:

(a) If $\Psi(\bar\nu) \geq 0$, then $\tilde\rho(\mathcal{M}) \geq \tilde\rho_2(\mathcal{M}) \geq \tilde\rho_1(\mathcal{M})$

(b) If $\Psi(\bar\nu) < 0$, then $\mathcal{Z}$ is nonempty and we have $\tilde\rho(\mathcal{M}) \geq \tilde\rho_3(\mathcal{M}) \geq \tilde\rho_1(\mathcal{M})$ and $\tilde\rho_3(\mathcal{M}) > \tilde\rho_2(\mathcal{M})$.

Computation of $\tilde{\rho}_3(\mathcal{M})$

Recall $\tilde{\rho}_3(\mathcal{M}) = \min_{\nu \in \mathcal{Z}} \prod_{i=1}^m s_1(A_i)^{\nu_i}$, $\tilde{\rho}_1(\mathcal{M}) = \min_{k=1,\dots,m} s_1(A_k)$ and $\Psi(\nu) = \sum_{i=1}^m \nu_i \log\left(\frac{\nu_i |\det A_i|}{s_1(A_i)^n}\right)$. Let

$$S \triangleq \{k \in \{1,\dots,m\} \mid s_1(A_k) = \tilde{\rho}_1(\mathcal{M})\}$$

Theorem 4 (M. 2022)

Assume we are in case (b) of Thm 2. If $s_1(A_1) = \cdots = s_1(A_m) = \tilde{\rho}_1(\mathcal{M})$ or if $\sum_{i \in S} |\det A_i|^{-1} \geq \tilde{\rho}_1(\mathcal{M})^{-n}$ then $\tilde{\rho}_3(\mathcal{M}) = \tilde{\rho}_1(\mathcal{M})$.
Otherwise, the min in $\tilde{\rho}_3(\mathcal{M})$ is attained at

$$\hat{\nu}_k(\beta) = \frac{s_1(A_k)^\beta |\det A_k|^{-1}}{\sum_{h=1}^m s_1(A_h)^\beta |\det A_h|^{-1}}$$

for some real value β. In this case $\tilde{\rho}_3(\mathcal{M}) > \tilde{\rho}_1(\mathcal{M})$ and $\tilde{\rho}_3(\mathcal{M})$ may be calculated numerically by solving the equations $\Psi(\hat{\nu}_1(\beta),\dots,\hat{\nu}_m(\beta)) = 0$ and substituting the corresponding solution(s) in $\tilde{\rho}_3(\mathcal{M})$.

Example

Let $\mathcal{M} = \{A_1, A_2\}$ with

$$A_1 = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}, \quad A_2 = \begin{pmatrix} c & 0 \\ 0 & c^{-1} \end{pmatrix}, \quad \text{with } c \in (0,1),\ \theta \in [0, 2\pi].$$

We have $s_1(A_1) = 1,\ s_2(A_2) = c,\ \det(A_1) = \det(A_2) = 1,\ \bar{\nu}_1 = \bar{\nu}_2 = \frac{1}{2}$ and $\Psi(\bar{\nu}) = \log\frac{1}{2} - \log c$.

Hence for $c \leq \frac{1}{2}$ we are in case (a) with lower bound $\tilde{\rho}_2(\mathcal{M}) = \frac{\sqrt{2}}{2}$.

For $c > \frac{1}{2}$ we are in case (b), the lower bound is $\tilde{\rho}_3(\mathcal{M}) = c^{1-\nu}$ where ν solves $\nu^{\frac{\nu}{\nu-1}}(\nu - 1) = c^2$

c	0.1	0.3	0.5	0.7	0.9
$\tilde{\rho}_1(\mathcal{M})$	0.1	0.3	0.5	0.7	0.9
$\tilde{\rho}_2(\mathcal{M})$	**0.7071**	**0.7071**	**0.7071**	0.7071	0.7071
$\tilde{\rho}_3(\mathcal{M})$	-	-	**0.7071**	**0.7613**	**0.9048**

What about regularity of $x_0 \mapsto \tilde{\rho}_{x_0}$?

Theorem 5 (Dettmann, Jungers & M. 2020)

Let $n = 2$ and assume that there exists $A \in \cup_{k\in\mathbb{N}} \mathcal{M}^k$ with nonreal eigenvalues and no power proportional to the identity, and that $\min_{i=1,\ldots,m} s_1(A_i)$ is only attained at a symmetric matrix not proportional to the identity. Then $x \mapsto \tilde{\rho}_x(\mathcal{M})$ is discontinuous everywhere in $\mathbb{R}^2$.

Idea of proof: A is a generalized irrational rotation, so its trajectories are dense (in angle) in positive and negative time. The symmetric matrix possesses an eigenvector with eigenvalue $\min_{i=1,\ldots,m} s_1(A_i)$. This eigenvector can be reached from a dense subset in which $\tilde{\rho}_x(\mathcal{M}) = \min_{i=1,\ldots,m} s_1(A_i)$. From Theorem 4 there exists a point s.t. $\tilde{\rho}_x(\mathcal{M}) > \min_{i=1,\ldots,m} s_1(A_i)$ and the same holds true along its dense orbit.

Examples

"Stanford-Urbano" example satisfies the conditions of Theorem 5 with generalized irrational rotation $A = A_1 A_2$

Consider the set of matrices $\mathcal{M} = \{A_1, A_2, A_3\}$ where

$$A_1 = \begin{pmatrix} -2 & 3 \\ -6 & 4 \end{pmatrix}, \quad A_2 = \begin{pmatrix} -0.8 & 0 \\ 0 & 2 \end{pmatrix}, \quad A_3 = \begin{pmatrix} 2 & -1 \\ -2 & -2 \end{pmatrix}.$$

The assumptions of Theorem 5 are satisfied since A_1 is a generalized irrational rotation, while the minimum singular value is equal to 0.8 and is attained by A_2. Theorems 2-4 give the lower bounds $\tilde{\rho}_2(\mathcal{M}) = 1.059$ and $\tilde{\rho}_3(\mathcal{M}) = 1.0675$ for $\tilde{\rho}(\mathcal{M})$.
Thus the system is stabilizable (exponentially) from a dense set of initial conditions and not stabilizable from another dense set.

Stabilization by periodic laws

Recall the joint spectral subradius

$$\check{\rho}(\mathcal{M}) = \lim_{k\to\infty} \inf_{M\in\mathcal{M}^k} \|M\|^{1/k} = \lim_{k\to\infty} \inf_{M\in\mathcal{M}^k} \rho(M)^{1/k}$$

The result below describes the relation between $\check{\rho}(\mathcal{M})$ and the existence of periodic stabilizing switching laws.

Theorem (M. 2022)

The following conditions are equivalent.

1. $\check{\rho}(\mathcal{M}) < 1$;
2. there exists a periodic switching law $\sigma : \mathbb{N} \to \{1, \dots, m\}$ s.t. $\lim_{k\to\infty} \prod_{i=0}^{k} A_{\sigma(i)} = 0$.

If, instead, $\check{\rho}(\mathcal{M}) \geq 1$ then, for a.e. $x \in \mathbb{R}^n$, there exists no periodic switching law $\sigma_x : \mathbb{N} \to \{1, \dots, m\}$ s.t. $\lim_{k\to\infty} \prod_{i=0}^{k} A_{\sigma_x(i)} x = 0$, i.e. the system is not stabilizable from x with a periodic switching law.

Sketch of the proof

The equivalence

$$\check{\rho}(\mathcal{M}) < 1 \iff \exists \sigma(\cdot) \text{ periodic s.t. } \lim_{k\to\infty} \prod_{i=0}^{k} A_{\sigma(i)} = 0$$

follows easily from the definitions.

Assume now $\check{\rho}(\mathcal{M}) \geq 1$. It follows easily that $\rho(A_{i_1} \cdots A_{i_k}) \geq 1$ for every matrix product $A_{i_1} \cdots A_{i_k} \in \mathcal{M}^k$, for every k.
$\Rightarrow \qquad A_{i_1} \cdots A_{i_k}$ has a stable subspace $V \subsetneq \mathbb{R}^n$
$\Rightarrow \qquad \cdots A_{i_k} A_{i_1} \cdots A_{i_k} x_0 \to 0 \iff x_0 \in V$
Since $\cup_k \mathcal{M}^k$ is countable we deduce that there exists a periodic stabilizing σ from x_0 only if x_0 belongs to a countable union $\cup_{\ell\in\mathbb{N}} V_\ell$ with $m(V_\ell) = 0$. Hence $m(\cup_{\ell\in\mathbb{N}} V_\ell) = 0$ concluding the proof.

Further remarks about periodic stabilization

As a consequence of the previous result:

Remark

If $\tilde{\rho}(\mathcal{M}) < 1 \leq \check{\rho}(\mathcal{M})$ then for every initial condition x_0 there exists a stabilizing switching law but, for a.e. x_0, such a switching law cannot be taken periodic.

This is not a pathological phenomenon. For instance if one takes

$$\mathcal{M} = \left\{ \frac{1}{\alpha} \begin{pmatrix} \cos\frac{\pi}{4} & \sin\frac{\pi}{4} \\ -\sin\frac{\pi}{4} & \cos\frac{\pi}{4} \end{pmatrix}, \frac{1}{\beta} \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 2 \end{pmatrix} \right\}$$

with $\alpha, \beta \in [0.9725, 1]$, then $\tilde{\rho}(\mathcal{M}) < 1 \leq \check{\rho}(\mathcal{M})$.

$\tilde{\rho}(\mathcal{M}) < \check{\rho}(\mathcal{M})$?

Partly motivated by the previous remark we would like to characterise the sets $\mathcal{M}$ s.t. $\tilde{\rho}(\mathcal{M}) < \check{\rho}(\mathcal{M})$. This problem appears to be highly nontrivial. In dimension $n = 2$ we have the following.

Theorem (M. 2022)

Assume that $n = 2$ and that there exists a coordinate transformation T such that

$$\mathcal{M}' = T\mathcal{M}T^{-1} = \{TMT^{-1} \mid M \in \mathcal{M}\}$$

satisfies

(a) $\exists A_1 \in \mathcal{M}'$ with $A_1 = \alpha R_\varphi$, where $\alpha \neq 0$ and R_φ is a rotation of an angle φ irrational with π;

(b) $|\det(A_1)| = \alpha^2 = \min_{A \in \mathcal{M}'} |\det(A)|$;

(c) $\exists A_2 \in \mathcal{M}'$ with real eigenvalues s.t. $s_1(A_2) < s_1(A_1) = |\alpha|$.

Then $\tilde{\rho}(\mathcal{M}) < \check{\rho}(\mathcal{M}) = |\alpha|$.

$\tilde{\rho}(\mathcal{M}) < \check{\rho}(\mathcal{M})$?

It is easier to study the analogous property for continuous-time systems. Consider the continuous-time switched system

$$\dot{x}(t) = B_{\sigma(t)}x(t), \quad B_\sigma \in \{B_1, \dots, B_m\} \subset M_n(\mathbb{R}) \tag{4}$$

The following result applies to discrete-time approximations of (4).

Theorem (M. 2022)

Assume that (4) is controllable in the projective space in finite time, i.e. $\exists T_0 > 0$ s.t. $\forall x_0 \in \mathbb{R}^n \setminus \{0\}$ and $\forall x_1 \in \mathbb{R}^n \setminus \{0\}$ there exists a trajectory $x(\cdot)$ of (4) satisfying $x(0) = x_0$ and $x(T) = \alpha x_1$ for some $T \in [0, T_0]$ and $\alpha \in \mathbb{R}$. Assume moreover that one of the matrices $B_1, \dots, B_m$ possesses an eigenvalue of real part r satisfying $r < \frac{1}{n} \min_{i=1,\dots,m} \mathrm{trace}(B_i)$[a].Then for any $\delta > 0$ small, setting $\mathcal{M} = \{e^{\delta B_1}, \dots, e^{\delta B_m}\}$ we have $\tilde{\rho}(\mathcal{M}) < \check{\rho}(\mathcal{M})$.

[a]verified unless the B_i with the eig. of min. real part is s.t. $Re(\lambda_j(B_i)) = r \ \forall j$

Questions/Perspectives

- What is $x_0 \mapsto \tilde{\rho}_{x_0}(\mathcal{M})$ for Stanford-Urbano example?
  Does it take a finite/discrete/continuous set of values?
- The bound of Theorems 3-4 may be improved by common linear transformation or by applying to $\mathcal{M}^k$ using $\tilde{\rho}(\mathcal{M}^k) = \tilde{\rho}(\mathcal{M})^k$.
  Does the lower bound for $\tilde{\rho}(\mathcal{M}^k)^{1/k}$ converges to $\tilde{\rho}(\mathcal{M})$?
- Generalisation of Thm 5: is the "dense discontinuity property" generic?
- What about continuous-time systems? For instance orbits are full-rank under generic conditions so Theorem 5 has probably no continuous-time counterpart
- How to better characterise the property $\tilde{\rho}(\mathcal{M}) < \check{\rho}(\mathcal{M})$.
  Is it generically satisfied for finite sets of matrices $\mathcal{M} \subset M_n(\mathbb{R})$?

Thank you for your attention!