Puzzles of converse Lyapunov theory for infinite-dimensional systems

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joint work with
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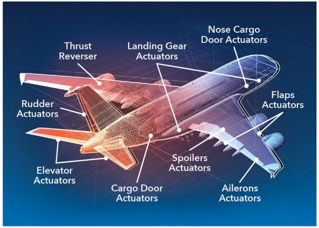
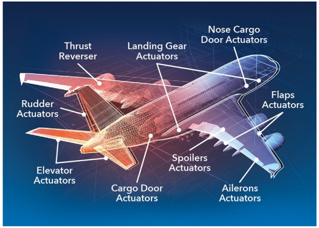
Infinite-dimensional systems everywhere



Aircraft control

Infinite-dimensional systems everywhere



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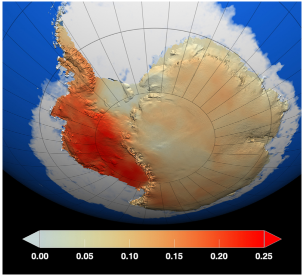
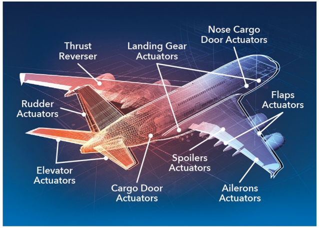


Climate models

Infinite-dimensional systems everywhere



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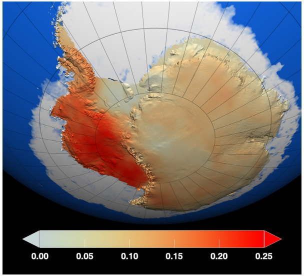

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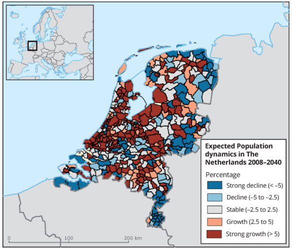
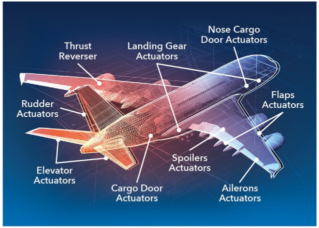


Population dynamics

Infinite-dimensional systems everywhere



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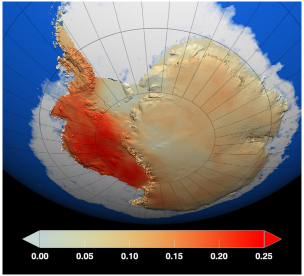

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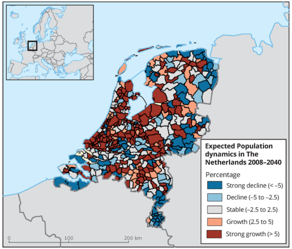

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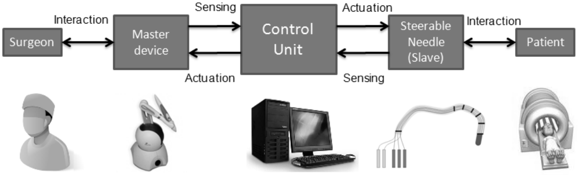
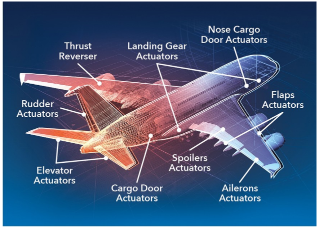


Human-robot-environment interaction



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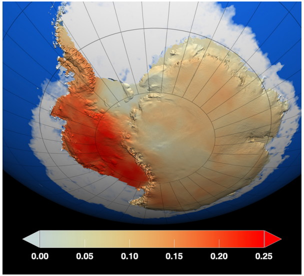

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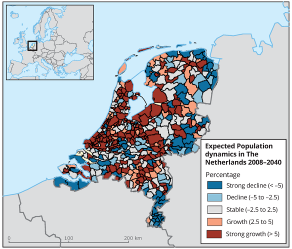

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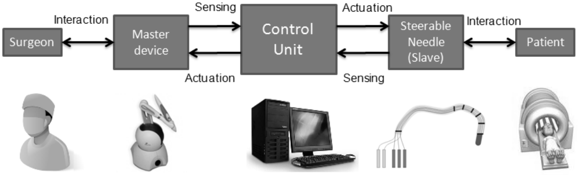

Human-robot-environment interaction

Chemical plant

Asymptotic properties and energy

Asymptotic properties of dynamical systems

- Forward-completeness
- Local stability
- Uniform boundedness of trajectories
- Global asymptotic stability

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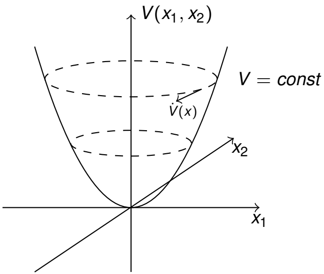

How to analyze these properties?

- It is hard to analyze these properties
- Mechanical systems: energy helps!
- Lyapunov functions are a mathematical abstraction of the physical energy.
- Above properties have criteria in terms of LFs

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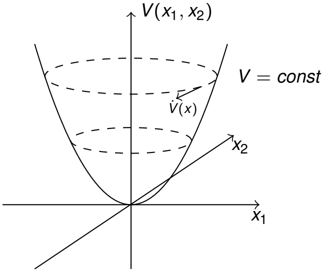

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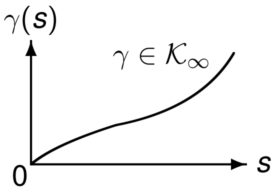
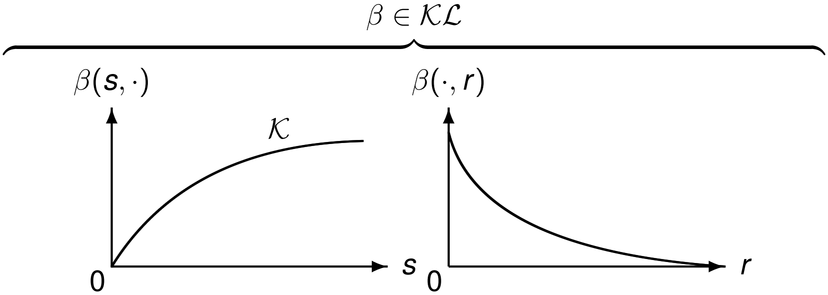
Today we revisit the very definition of a Lyapunov function.

Outline

Comparison functions





$$\dot{x} = Ax + f(x), \quad x(t) \in X. \qquad \text{(SEE)}$$

Here:

- X is a Banach space
- $A : D(A) \subset X \to X$ generates a strongly continuous semigroup $(T(t))_{t\geq 0}$ on X, i.e.,
$$\dot{x} = Ax$$
is well-posed, with a solution $t \mapsto T(t)x_0$ for $x(0) = x_0$.
- $f : X \to X$ is Lipschitz continuous on bounded balls.

Definition (Mild solutions)

$x \in C([0,\tau], X)$ is called a mild solution of (SEE) on $[0,\tau]$ corresponding to certain $x_0 \in X$, if x solves the integral equation

$$x(t) = T(t)x_0 + \int_0^t T(t-s)f(x(s))\,ds.$$

$$\dot{x} = Ax + f(x), \quad x(t) \in X. \tag{SEE}$$

Example 1

$$A \in L(X) \quad \Rightarrow \quad T(t) = e^{tA} = \sum_{k=0}^{\infty} \frac{A^k t^k}{k!}, \quad t \geq 0.$$

Mild solution of (SEE):

$$x(t) = e^{tA}x_0 + \int_0^t e^{(t-s)A} f\big(x(s)\big)\,ds.$$

Example 2: Heat equation

$$x_t(z,t) = x_{zz}(z,t), \quad z \in (0,\pi), \quad t > 0,$$

with Dirichlet boundary conditions

$$x(0,t) = x(\pi,t) = 0, \quad t \geq 0.$$

Then

$$T(t)x_0 := \sum_{k=1}^{\infty} e^{-k^2 t} \langle x_0, \phi_k \rangle_X \phi_k, \quad x_0 \in X, \quad t \geq 0,$$

$$\phi_k : z \mapsto \sqrt{\frac{2}{\pi}} \sin(kz), \quad k \in \mathbb{N}, \quad z \in (0,\pi).$$

$$\dot{x} = Ax + f(x), \quad x(t) \in X. \tag{SEE}$$

Definition (UGAS)

(SEE) is called uniformly globally asymptotically stable (UGAS) if there is $\beta \in \mathcal{KL}$:

$$\|\phi(t,x)\|_X \leq \beta(\|x\|_X, t), \quad x \in X, \quad t \geq 0.$$

Definition (Lyapunov function)

$V \in C(X, \mathbb{R}_+)$ is a (coercive) Lyapunov function if $\exists \psi_1, \psi_2, \alpha \in \mathcal{K}_\infty$:

- $\psi_1(\|x\|_X) \leq V(x) \leq \psi_2(\|x\|_X) \quad \forall x \in X,$
- $\dot{V}(x) := \limsup_{h\to 0} \frac{1}{h}\Big(V(\phi(h,x)) - V(x)\Big) \leq -\alpha(\|x\|_X) \quad \forall x \in X.$

Theorem (Henry, 1981)

(SEE) is UGAS $\Leftrightarrow$ there is a Lipschitz continuous coercive Lyapunov function.

Lyapunov view on Datko's theorem

$$\dot{x} = Ax \tag{LIN}$$

Theorem (Datko, 1970)

Let X be a Hilbert space.
(LIN) is exponentially stable $\Leftrightarrow$ (LIN) is UGAS $\Leftrightarrow$ $\exists$ positive self-adjoint $P \in L(X)$ s.t.

$$\langle Ax, Px\rangle + \langle Px, Ax\rangle = -\|x\|_X^2, \quad x \in D(A). \tag{LE}$$

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- Define $V(x) := \langle Px, x\rangle$.
- Then

$$0 < V(x) \leq \|P\|\|x\|_X^2, \quad x \neq 0,$$

  and

$$\dot{V}(x) = \langle Ax, Px\rangle + \langle Px, Ax\rangle = -\|x\|_X^2.$$

- However, there may be no $\psi_1 \in \mathcal{K}_\infty$: $\quad V(x) \geq \psi_1(\|x\|_X), \quad x \in X$.

If $A = A^*$, then the solution of (LE) is $P = -\frac{1}{2}A^{-1}$, and if $\sigma(A)$ is unbounded, then $0 \in \sigma(P)$.

Lyapunov view on Pazy's theorem

$$\dot{x} = Ax, \quad x(t) \in X. \tag{LIN}$$

Theorem (Pazy, 1983)

(LIN) is exponentially stable $\quad \Leftrightarrow \quad \exists p \in [1, +\infty)$: $\int_0^\infty \|T(t)x\|_X^p dt < \infty \quad \forall x \in X.$

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Theorem (Pazy, 1983)

(LIN) is exponentially stable $\Leftrightarrow$ $\exists p \in [1, +\infty)$: $V(x) := \int_0^\infty \|T(t)x\|_X^p dt < \infty \quad \forall x \in X$.

Let us analyse V:

- $0 < V(x) \leq a\|x\|_X^p$, for some $a > 0$ and all $x \in X$.
- V is continuous.
- Dissipation:

$$\begin{aligned}\dot{V}(x) &= \limsup_{h\to 0} \frac{V(T(h)x) - V(x)}{h} = \limsup_{h\to 0} \frac{1}{h}\Big(\int_0^\infty \|T(t+h)x\|_X^p dt - \int_0^\infty \|T(t)x\|_X^p dt\Big) \\ &= \limsup_{h\to 0} -\frac{1}{h}\int_0^h \|T(t)x\|_X^p dt \\ &= -\|x\|_X^p.\end{aligned}$$

$$\dot{x} = Ax, \quad x(t) \in X. \tag{LIN}$$

Theorem (Littman, 1989)

(LIN) is exponentially stable $\quad\Leftrightarrow\quad \exists \alpha \in \mathcal{K}\text{:} \quad \int_0^\infty \alpha(\|T(t)x\|_X)dt < \infty \;\; \forall x \in X.$

Lyapunov view on Littman's theorem

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Moral

The classical criteria of exp. stability due to Datko, Pazy, Littman are the results about non-coercive Lyapunov functions.

Nonlinear vs. linear theory

Nonlinear theory tells:

$$\dot{x} = Ax + f(x), \quad x(t) \in X. \tag{SEE}$$

(SEE) is UGAS $\quad\Leftrightarrow\quad$ there is a coercive Lyapunov function.

Linear theory tells:

$$\dot{x} = Ax, \quad x(t) \in X. \tag{LIN}$$

(LIN) is UGAS $\quad\Leftrightarrow\quad$ there is a non-coercive Lyapunov function of particular type.

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Is coercivity needed?

$$\dot{x} = Ax + f(x), \quad x(t) \in X. \tag{SEE}$$

Definition (Non-coercive Lyapunov function)

$V \in C(X, \mathbb{R}_+)$ is a non-coercive Lyapunov function if $\exists \psi_2, \alpha \in \mathcal{K}_\infty$:

- $0 < V(x) \leq \psi_2(\|x\|_X) \quad \forall x \in X \setminus \{0\}$,
- $\dot{V}(x) \leq -\alpha(\|x\|_X) \quad \forall x \in X$.

Direct non-coercive Lyapunov theorems

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Theorem (Direct nonlinear non-coercive Lyapunov theorem (AM, Wirth; JDE 2019))

(SEE) is UGAS $\Leftrightarrow$ *The following holds:*

- *There is a non-coercive Lyapunov function for (SEE).*
- *The flow map ϕ is continuous at equilibrium.*
- *For any $\tau > 0$ and any bounded $B \subset X$ it holds that $\phi([0, \tau], B)$ is a bounded set.*

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Proof.

- Proof via Banach-Steinhaus theorem is not possible (no linearity)
- Proof by comparison principles is not possible (no coercivity)
- Radically new strategy was needed, based on the characterizations of UGAS in terms of uniform weak attractivity, see (AM, Wirth; JDE 2019) and (AM, Wirth; MCSS 2019).

$\square$

Remark

For (SEE) ϕ is continuous at 0, but this condition cannot be dropped for more general

Direct non-coercive Lyapunov theorems for linear systems

Theorem (M., Wirth, JDE 2019, a corollary of a general nonlinear result)

$$\dot{x} = Ax, \quad x(t) \in X. \tag{LIN}$$

TFAE:

- *(LIN) is exponentially stable*
- *There is a quadratic non-coercive Lyapunov function*
- *There is a non-coercive Lyapunov function*
- *There is a coercive Lyapunov function*

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What about coercive quadratic Lyapunov functions?

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Theorem (AM, F. Schwenninger; submitted to MCSS, 2023)

Let X be a Hilbert space, and let (LIN) be exponentially stable. TFAE:

1. *There is a coercive, quadratic Lyapunov function for (LIN).*
2. *T is similar to a contraction semigroup,*
   i.e. there exists a boundedly invertible operator $S : X \to X$ so that $(ST(t)S^{-1})_{t\geq 0}$ is a contraction semigroup.

If T is analytic, above conditions are equivalent to

4. *The function $V : D(A) \to \mathbb{R}_+, \quad x \mapsto \int_0^\infty \|(-A)^{\frac{1}{2}} T(t)x\|^2 \, \mathrm{d}t$*
   extends to a coercive, quadratic Lyapunov function from X to $\mathbb{R}_+$ for the system (LIN).

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Proposition (AM, F. Schwenninger; submitted to MCSS, 2023)

For any ∞-dim Hilbert space X there is a generator A of an exponentially stable, analytic semigroup T on X s.t. (LIN) has no coercive, quadratic LF.

What about coercive quadratic Lyapunov functions?

$$\dot{x} = Ax, \quad x(t) \in X. \tag{LIN}$$

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- It is not evident, that the integral in V converges.
- Here we essentially use that T is similar to contraction semigroup.

If $A = A^*$, our LF takes form

$$
\begin{aligned}
V(x) = \int_0^\infty \|(-A)^{\frac{1}{2}} T(t)x\|^2 dt &= \int_0^\infty \left\langle (-A)^{\frac{1}{2}} T(t)x, (-A)^{\frac{1}{2}} T(t)x \right\rangle dt \\
&= \int_0^\infty \langle (-A) T(2t)x, x \rangle \, dt \\
&= \left\langle \int_0^\infty (-A) T(2t)x dt, x \right\rangle \\
&= \frac{-1}{2} \left\langle \int_0^\infty A T(t)x dt, x \right\rangle \\
&= \frac{1}{2} \langle x, x \rangle = \frac{1}{2} \|x\|_X^2.
\end{aligned}
$$

Remark (If $A = A^*$, there is also a non-coercive LF for (LIN))

$$W(x) = \tfrac{1}{2} \left\langle (-A)^{-1} x, x \right\rangle, \; x \in X.$$

Take-Home Message I

Results for $\dot{x} = Ax + f(x)$

- UGAS $\Leftrightarrow$ $\exists$ coercive LF
- UGAS $\Leftrightarrow$ The following holds:
  - $\exists$ non-coercive LF.
  - The flow map ϕ is continuous at equilibrium.
  - For any $\tau > 0$ and any bounded $B \subset X$ it holds that $\phi([0, \tau], B)$ is a bounded set.

Results for $\dot{x} = Ax$

- $\exists$ non-coercive, quadratic LF $\Leftrightarrow$ $\exists$ non-coercive LF $\Leftrightarrow$ exp. stability
- $\exists$ coercive, quadratic LF $\Leftrightarrow$ T is exp. stable & similar to a contraction

Literature

- AM, Wirth. *Non-coercive Lyapunov functions for infinite-dimensional systems*, JDE, 2019.
- AM, Wirth. *Existence of non-coercive Lyapunov functions is equivalent to integral uniform global asymptotic stability*, MCSS, 2019.
- AM, Schwenninger. *Coercive quadratic converse ISS Lyapunov theorems for linear analytic systems*, Submitted to MCSS, 2023.

Systems with inputs

Definition

The triple $\Sigma = (X, \mathcal{U}, \phi)$, $\quad \phi : \mathbb{R}_+ \times X \times \mathcal{U} \to X$ is called control system, if:

(Σ1) Forward-completeness: for every $x \in X$, $u \in \mathcal{U}$ and for all $t \geq 0$ the value $\phi(t, x, u) \in X$ is well-defined.

(Σ2) Continuity: for each $(x, u) \in X \times \mathcal{U}$ the map $t \mapsto \phi(t, x, u)$ is continuous.

(Σ3) Cocycle property: for all $t, h \geq 0$, for all $x \in X$, $u \in \mathcal{U}$ we have

$$\phi(h, \phi(t, x, u), u(t + \cdot)) = \phi(t + h, x, u).$$

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Examples

- Ordinary differential equations
- Evolution PDEs with Lipschitz nonlinearities
- Broad classes of boundary control systems
- Time-delay systems
- Heterogeneous systems with distinct components

Definition (Sontag, 1989, for ODEs)

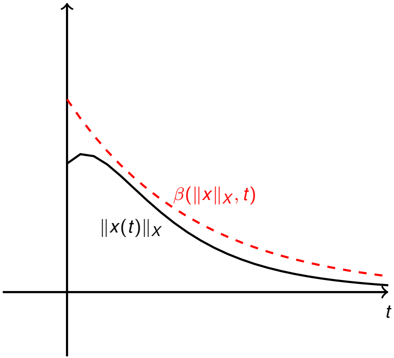
$$\text{ISS} \quad :\Leftrightarrow \quad \|x(t)\|_X \le \beta(\|x\|_X, t) + \gamma(\|u\|_{\mathcal{U}}), \quad \forall x, t, u.$$

increasing in $\|x\|_X$

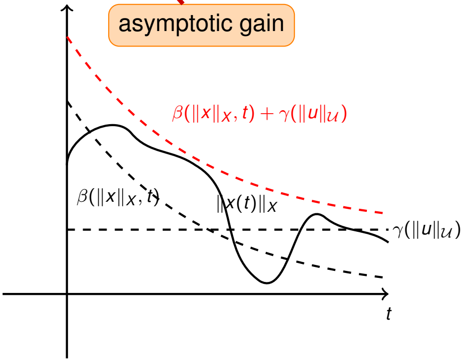
decreasing to 0 in t

$\gamma(0) = 0$, increasing

asymptotic gain



(a) $u \equiv 0$



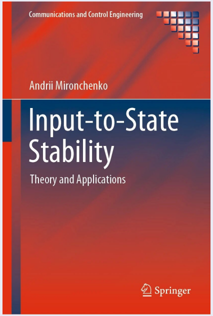
(b) $u \not\equiv 0$

Why ISS?

1. Unified theory of internal and external stability
2. Robust control (stabilization, tracking) of nonlinear systems
3. Control in presence of quantizations, delays, unmodeled dynamics, etc.
4. Robust observer design for nonlinear systems
5. Analysis & control of large-scale networks
6. ...

Overview of ISS theory

- AM. *Input-to-State Stability*, Springer, 2023.
- Chaillet, Karafyllis, Pepe, Wang. *The ISS framework for time-delay systems: a survey*, MCSS, 2023.
- AM, Prieur. Input-to-state stability of infinite-dimensional systems: recent results and open questions, *SIAM Review*, 2020.
- Karafyllis, Krstic. *Input-to-State Stability for PDEs*, Springer, 2019.
- Dashkovskiy, Efimov, Sontag. *Input to state stability and allied system properties*, 2011.
- Sontag. *Input to state stability: Basic concepts and results*, 2008.



ISS Lyapunov functions

Definition (Sontag, 1989, for ODEs)

$$\text{ISS} \quad :\Leftrightarrow \quad \|x(t)\|_X \leq \beta(\|x\|_X, t) + \gamma(\|u\|_{\mathcal{U}}), \quad \forall x, t, u.$$

Definition (ISS Lyapunov functions)

Let X be a Banach space, $\mathcal{U} := PC(\mathbb{R}_+, U)$, and $\Sigma := (X, \mathcal{U}, \phi)$ be a control system.
$V \in C(X, \mathbb{R}_+)$ is a (coercive) ISS Lyapunov function iff $\exists \psi_1, \psi_2, \sigma, \alpha \in \mathcal{K}_\infty$:

$$\psi_1(\|x\|_X) \leq V(x) \leq \psi_2(\|x\|_X) \quad \forall x \in X,$$

$$\dot{V}_u(x) \leq -\alpha(\|x\|_X) + \sigma(\|u\|_{\mathcal{U}}) \quad \forall x \in X, \ \forall u \in \mathcal{U}.$$

Lyapunov theory for boundary control systems: Overview

Develop methods for ISS analysis of nonlinear PDEs with boundary inputs

Lyapunov methods: main questions

- Direct Lyapunov theorems
- Converse Lyapunov theorems
- Construction of Lyapunov functions

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Good news I

- $\exists$ *coercive* ISS LF $\Rightarrow$ ISS.
- [M., Wirth, SCL, 2018], based on [Sontag, Wang, SCL, 1995]

  Let f be Lipschitz on bounded balls in $X \times U$. Then

$$\dot{x} = Ax + f(x, u) \tag{NL}$$

  is ISS $\quad \Leftrightarrow \quad \exists$ ISS-LF.

Develop methods for ISS analysis of nonlinear PDEs with boundary inputs

Lyapunov methods: main questions

- Direct Lyapunov theorems
- Converse Lyapunov theorems
- Construction of Lyapunov functions

Good news II

- Some motivating results exploiting quadratic Lyapunov functions appeared:
  - Tanwani, Prieur, and Tarbouriech. *Disturbance-to-state stabilization and quantized control for linear hyperbolic systems*, 2017.
  - Zheng, Zhu. *Input-to-state stability with respect to boundary disturbances for a class of semi-linear parabolic equations.* Automatica, 2018.
  - Schwenninger. *Input-to-state stability for parabolic boundary control: linear and semilinear systems, 2020.*

Lyapunov theory for boundary control systems: Overview

Develop methods for ISS analysis of nonlinear PDEs with boundary inputs

Lyapunov methods: main questions

- Direct Lyapunov theorems
- Converse Lyapunov theorems
- Construction of Lyapunov functions

Distressing (yet motivating) news

- Classical "coercive" Lyapunov methods failed to address the ISS of linear heat equation with Dirichlet boundary inputs.

$$\begin{aligned} x_t(z,t) &= x_{zz}(z,t), \quad z \in (0,1), \ t > 0, \\ x(0,t) &= 0, \quad x(1,t) = u(t), \quad t > 0. \end{aligned}$$

- What about converse ISS Lyapunov theorems for boundary control systems?

Lyapunov theory for boundary control systems: Overview

Develop methods for ISS analysis of nonlinear PDEs with boundary inputs

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- Direct Lyapunov theorems
- Converse Lyapunov theorems
- Construction of Lyapunov functions

Thriving of non-Lyapunov methods

- Karafyllis, Krstic. *ISS with respect to boundary disturbances for 1-D parabolic PDEs.* IEEE TAC, 2016.
- Jacob, Nabiullin, Partington, Schwenninger. *Infinite-dimensional input-to-state stability and Orlicz spaces.* SICON, 2018.
- M., Karafyllis, Krstic. *Monotonicity methods for input-to-state stability of nonlinear parabolic PDEs with boundary disturbances.* SICON, 2019.
- Zheng, Zhu. *A De Giorgi iteration-based approach for the establishment of ISS properties for Burgers' equation with boundary and in-domain disturbances.* IEEE TAC, 2019.

Let X be a Banach space, $\mathcal{U} := PC(\mathbb{R}_+, U)$, and $\Sigma := (X, \mathcal{U}, \phi)$ be a control system.

Definition

$V : X \to \mathbb{R}_+$ is a non-coercive ISS-Lyapunov function iff $\exists \psi_2, \sigma, \alpha \in \mathcal{K}_\infty$:

(i) $0 < V(x) \leq \psi_2(\|x\|_X) \quad \forall x \neq 0,$

(ii) $\dot{V}_u(x) \leq -\alpha(\|x\|_X) + \sigma(\|u(0)\|_U) \quad \forall x \in X, \forall u \in \mathcal{U}.$

Lyapunov methods: main questions

- Direct Lyapunov theorems
- Converse Lyapunov theorems
- Construction of Lyapunov functions

Let X be a Banach space, $\mathcal{U} := PC(\mathbb{R}_+, U)$, and $\Sigma := (X, \mathcal{U}, \phi)$ be a control system.

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Theorem (Direct nonlinear non-coercive ISS Lyapunov theorem)

Let Σ possess bounded finite-time reachability sets and let ϕ be continuous at 0.

If $\exists$ a non-coercive ISS Lyapunov function for $\Sigma \quad \Rightarrow \quad \Sigma$ is ISS.

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If $\exists$ a non-coercive ISS Lyapunov function for $\Sigma \quad \Rightarrow \quad \Sigma$ is ISS.

- We cannot use “linear” methods
- We cannot use comparison principle

Let X be a Banach space, $\mathcal{U} := PC(\mathbb{R}_+, U)$, and $\Sigma := (X, \mathcal{U}, \phi)$ be a control system.

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$V : X \to \mathbb{R}_+$ is a non-coercive ISS-Lyapunov function iff $\exists \psi_2, \sigma, \alpha \in \mathcal{K}_\infty$:

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Shown in

- Jacob, M., Partington, Wirth. *Noncoercive Lyapunov functions for input-to-state stability of infinite-dimensional systems.* SICON, 2020.

A proof is based on characterizations of ISS, obtained in

- M. *Local input-to-state stability: Characterizations and counterexamples.* Sys. & Con. Lett., 2016.
- M., Wirth. *Characterizations of input-to-state stability for infinite-dimensional systems.* IEEE TAC, 2018.

Let X be a Banach space, $\mathcal{U} := PC(\mathbb{R}_+, U)$, and $\Sigma := (X, \mathcal{U}, \phi)$ be a control system.

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Theorem (Direct nonlinear non-coercive ISS Lyapunov theorem)

Let Σ possess bounded finite-time reachability sets and let ϕ be continuous at 0.

If $\exists$ a non-coercive ISS Lyapunov function for $\Sigma \quad \Rightarrow \quad \Sigma$ is ISS.

How to construct non-coercive ISS Lyapunov functions?

Let us look closer at the linear systems with boundary inputs

$$\dot{x}(t) = Ax(t) + Bu(t), \quad x(0) = x_0 \in X, t > 0. \tag{1}$$

- A generates a C_0-semigroup $(T(t))_{t\geq 0}$ on a Banach space X.
- $B \in L(U, X_{-1})$ for some Banach space U.
- X_{-1} is the completion of X w.r.t. $\|x\|_{X_{-1}} := \|(\beta - A)^{-1}x\|_X$, for some $\beta \in \rho(A)$.
- T extends uniquely to T_{-1} on X_{-1} whose generator A_{-1} is an extension of A.
- (1) is well-posed on X_{-1}: $\forall x_0 \in X$ and $\forall u \in L^1_{\mathrm{loc}}([0,\infty), U)$, the function $\phi(\cdot, x_0, u) : [0,\infty) \to X_{-1}$,

$$\phi(t, x_0, u) := T(t)x_0 + \int_0^t T_{-1}(t-s)Bu(s)ds, \quad t \geq 0,$$

is called mild solution of (1).

Linear systems with admissible control operators

Let us look closer at the linear systems with boundary inputs

$$\dot{x}(t) = Ax(t) + Bu(t), \quad x(0) = x_0 \in X, t > 0. \tag{1}$$

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$$\phi(t, x_0, u) := T(t)x_0 + \int_0^t T_{-1}(t-s)Bu(s)ds, \quad t \geq 0,$$

is called mild solution of (1).

That's great, but the trajectory is now in X_{-1}

(Global) well-posedness for L^q-inputs

- $B \in L(U, X_{-1})$ is called a q-admissible control operator for $(T(t))_{t\geq 0}$, where $1 \leq q \leq \infty$, if

$$t \geq 0, \quad x_0 \in X, \quad u \in L^q([0,\infty), U) \quad \Rightarrow \quad \phi(t, x_0, u) \in X.$$

$$\Sigma : \dot{x} = Ax + Bu.$$

Theorem (Jacob, Nabiullin, Partington, Schwenninger, SICON, 2018)

Σ is L^q-ISS $\quad\Leftrightarrow\quad$ T is exponentially stable $\quad\wedge\quad$ B is q-admissible.

Theorem (Jacob, AM, Partington, Wirth, SICON, 2020)

$$\Sigma : \dot{x} = Ax + Bu.$$

Let A generate a C_0-semigroup $(T(t))_{t\geq 0}$ on a Hilbert space X.
Assume that there is $P \in L(X)$ satisfying:

(i) *P satisfies* $\mathrm{Re}\,\langle Px, x\rangle_X > 0, \quad x \in X\backslash\{0\}$.

(ii) *P satisfies Lyapunov inequality*

$$\mathrm{Re}\,\langle (PA + A^*P)x, x\rangle_X \leq -\langle x, x\rangle_X\,, \quad x \in D(A), \tag{2}$$

(iii) *It holds that $Im(P) \subset D(A^*)$.*

(iv) *PA is bounded.*

Then

$$V(x) := \mathrm{Re}\,\langle Px, x\rangle_X \tag{3}$$

is a non-coercive ISS Lyapunov function for (1) with any ∞-admissible input operator $B \in L(U, X_{-1})$, and thus (1) is ISS for such B.

Self-adjoint A

Proposition (Jacob, AM, Partington, Wirth, SICON, 2020)

$$\Sigma : \dot{x} = Ax + Bu.$$

Let:

- *A generate an exponentially stable C_0-semigroup $(T(t))_{t\geq 0}$ on a complex Hilbert space X*
- *$B \in L(U, X_{-1})$ be ∞-admissible.*
- *$A = A^*$.*

Then

$$V(x) := -\mathrm{Re}\, \langle A^{-1}x, x\rangle_X \tag{4}$$

is an ISS Lyapunov function.

Heat equation with Neumann boundary control

$$\begin{aligned} &x_t(z,t) = x_{zz}(z,t), \quad z \in (0,1),\ t > 0,\\ &x(0,t) = 0, \quad x_z(1,t) = u(t), \quad t > 0,\\ &x(z,0) = x_0(z). \end{aligned}$$

We choose $X = L^2(0,1)$, $U = \mathbb{C}$, $\quad Af := f''$, $\qquad f \in D(A) := \left\{ f \in H^2(0,1) \mid f(0) = f'(1) = 0 \right\}.$

Consider $V(x) := \int_0^1 x^2(z)dz$.

$$\begin{aligned} \frac{d}{dt}V(x) := \int_0^1 2x(z)x_{zz}(z)dz \quad &= \quad 2x(1)u(0) - 2\int_0^1 x_z^2(z)dz \leq \varepsilon|x(1)|^2 + \frac{1}{\varepsilon}|u(0)|^2 - 2\int_0^1 x_z^2(z)dz\\ &\leq \quad \varepsilon\int_0^1 x^2(z)dz + \varepsilon\int_0^1 x_z^2(z)dz + \frac{1}{\varepsilon}|u(0)|^2 - 2\int_0^1 x_z^2(z)dz\\ &\leq \quad \varepsilon\int_0^1 x^2(z)dz - (2-\varepsilon)\frac{\pi^2}{4}\int_0^1 x^2(z)dz + \frac{1}{\varepsilon}|u(0)|^2\\ &= \quad -\left(\frac{\pi^2(2-\varepsilon)}{4} - \varepsilon\right)V(x) + \frac{1}{\varepsilon}|u(0)|^2. \end{aligned}$$

From the last inequality we obtain L_2-ISS.

Heat equation with Dirichlet boundary control

$$
\begin{aligned}
x_t(z,t) &= x_{zz}(z,t), \quad z \in (0,1), \ t > 0, \\
x(0,t) &= 0, \quad x(1,t) = u(t), \quad t > 0
\end{aligned}
$$

We choose $X = L^2(0,1)$, $U = \mathbb{C}$,

$$
Af := f'', \qquad f \in D(A) := \left\{ f \in H^2(0,1) \mid f(0) = f(1) = 0 \right\}.
$$

Consider again $V(x) := \int_0^1 x^2(z)dz$.

$$
\frac{d}{dt}V(x) := \int_0^1 2x(z)x_{zz}(z)dz = 2x_z(1)u(0) - 2\int_0^1 x_z^2(z)dz \leq 2x_z(1)u(0) - 2\pi V(x).
$$

No constructions of coercive ISS Lyapunov functions are available for this system.

Heat equation with Dirichlet boundary control

$$x_t(z,t) = x_{zz}(z,t), \quad z \in (0,1),\ t > 0,$$
$$x(0,t) = 0, \quad x(1,t) = u(t), \quad t > 0$$

We choose $X = L^2(0,1)$, $U = \mathbb{C}$,

$$Af := f'', \qquad f \in D(A) := \left\{ f \in H^2(0,1) \mid f(0) = f(1) = 0 \right\}.$$

This system is ISS for $X := L_2(0,1)$ and $\mathcal{U} := L_\infty(\mathbb{R}_+, \mathbb{R})$, as shown via:

✓ Admissibility approach

- Jacob, Nabiullin, Partington, Schwenninger. *Infinite-dimensional input-to-state stability and Orlicz spaces.* SICON, 2018.

✓ Spectral analysis

- Karafyllis, Krstic. *ISS with respect to boundary disturbances for 1-D parabolic PDEs.* IEEE TAC, 2016.

✓ Monotonicity methods

- M., Karafyllis, Krstic. *Monotonicity methods for input-to-state stability of nonlinear parabolic PDEs with boundary disturbances.* SICON, 2019.

Heat equation with Dirichlet boundary control

$$Af := f'', \qquad f \in D(A) := \qquad \left\{ f \in H^2(0,1) \mid f(0) = f(1) = 0 \right\}, \qquad B = \delta_1'.$$

- A generates an exponentially stable analytic C_0-semigroup on X
- $B \in X_{-1} = L(U, X_{-1})$ is ∞-admissible
- $A = A^*$

By previous results, the system is L_∞-ISS and the corresponding non-coercive ISS Lyapunov function is:

$$V(x) = -\langle A^{-1}x, x\rangle_X = \int_0^1 \left(\int_z^1 (z-\tau)x(\tau)d\tau \right) \overline{x(z)}dz.$$

Coercive quadratic ISS LFs for linear analytic systems

Let X be a Hilbert space.

Let A generate an analytic semigroup over a Banach space X and let $\alpha > 0$.

$$X_{-\alpha} \quad := \quad \text{the completion of } X \text{ with respect to the norm } x \mapsto \|(aI - A)^{-\alpha} x\|_X.$$

Theorem (AM, Schwenninger, submitted to MCSS 2023)

- *Let A generate an exponentially stable analytic semigroup T similar to a contraction.*
- *Let $B \in L(U, X_{-p})$ for some $p < \frac{1}{2}$.*

Then:

$$V : D(A) \to \mathbb{R}_+, \quad x \mapsto \int_0^\infty \|(-A)^{\frac{1}{2}} T(t) x\|^2 dt$$

(extended to X) is a coercive quadratic L^2-ISS Lyapunov function for

$$\dot{x} = Ax + Bu.$$

In particular, this system is L^2-ISS.

Coercive quadratic ISS LFs for linear analytic systems

Theorem (AM, Schwenninger, submitted to MCSS 2023)

- *Let A generate an exponentially stable analytic semigroup T similar to a contraction.*
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$$\dot{x} = Ax + Bu.$$

In particular, this system is L^2-ISS.

Theorem (Jacob, Nabiullin, Partington, Schwenninger, SICON, 2018)

Σ *is L^2-ISS* $\Leftrightarrow$ *T is exponentially stable* $\wedge$ *B is 2-admissible.*

Coercive quadratic ISS LFs for linear systems with $A = A^*$

Theorem (AM, Schwenninger, submitted to MCSS 2023)

- *Let $A = A^*$ generate an exponentially stable analytic semigroup.*
- *Let $B \in L(U, X_{-\frac{1}{2}})$.*

Then:

$$V(x) := \frac{1}{2}\|x\|^2$$

is a coercive quadratic L^2-ISS Lyapunov function for

$$\dot{x} = Ax + Bu.$$

In particular, this system is L^2-ISS.

Theorem (Jacob, Nabiullin, Partington, Schwenninger, SICON, 2018)

$$\Sigma \text{ is } L^2\text{-ISS} \quad \Leftrightarrow \quad T \text{ is exponentially stable} \quad \wedge \quad B \text{ is 2-admissible.}$$

Conclusion

We discussed

- Coercive ISS Lyapunov theorems
- Non-coercive ISS Lyapunov theorems
- Lyapunov theory for linear systems with admissible input operators
- Conditions for existence of coercive quadratic L^2-ISS Lyapunov function.

References

- AM, Wirth. *Non-coercive Lyapunov functions for infinite-dimensional systems.* JDE, 2019.
- AM, Wirth. *Existence of non-coercive Lyapunov functions is equivalent to integral uniform global asymptotic stability*, MCSS, 2019.
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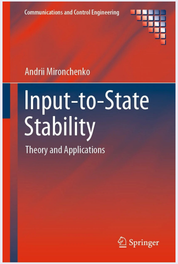

Thank you for Your attention!

We discussed

- Coercive ISS Lyapunov theorems
- Non-coercive ISS Lyapunov theorems
- Lyapunov theory for linear systems with admissible input operators
- Conditions for existence of coercive quadratic L^2-ISS Lyapunov function.

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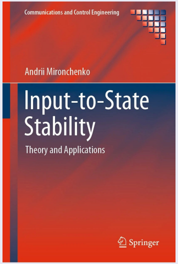

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