Stabilization of evolution systems

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Introduction

The stabilization problem starts like any other control problem:

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Goal: for any initial condition the system is stable and

$$\|z(t, \cdot)\|_X \to 0$$

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Goal: exponential stability

$$\|z(t,\cdot)\|_X \leq Ce^{-\gamma t}\|z(0,\cdot)\|_X, \quad \forall\, t \in [0, +\infty).$$

Examples

The Saint-Venant equations

$$\begin{aligned}
&\partial_t A + \partial_x (AV) = 0,\\
&\partial_t V + \partial_x \left(\frac{V^2}{2} + gL(A,x)\right) - \underbrace{S_b(x)}_{\text{slope}} + \underbrace{S(A,V,x)}_{\text{friction}} = 0.
\end{aligned}$$

(Boundary) feedback controls

$$v(t,0) = G_1(h(t,0)), \quad v(t,L) = G_2(h(t,L))$$

Theorem (A.H., Shang 2019)

The system is (locally) exponentially stable for the H^2 norm if

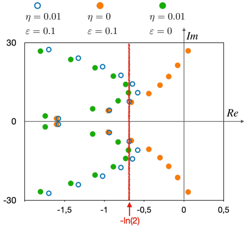
$$\begin{aligned}
G_1'(0) &\in \left(-\frac{g\partial_A G(A^*(0),0)}{V^*(0)}, -\frac{V^*(0)}{A^*(0)}\right),\\
G_2'(0) &\in \mathbb{R} \setminus \left[-\frac{g\partial_A G(A^*(L),L)}{V^*(L)}, -\frac{V^*(L)}{A^*(L)}\right].
\end{aligned}$$

Introduction

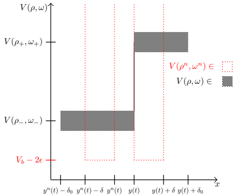
Stabilization: a very useful problem in practice

- Always some perturbations in reality
- Technologies' complexity is increasing
- Automation $\rightarrow$ stabilization

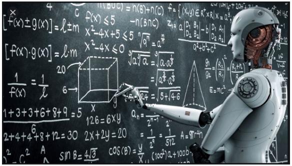
Outline of the talk



1. Robustness and hyperbolic systems



2. Control of traffic flows



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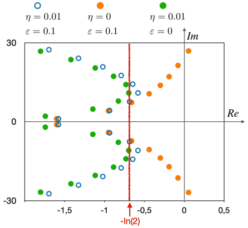

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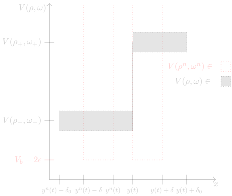

2. Control of traffic flows

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Motivation

Consider the following control problem

$$\begin{cases} \partial_t y_1 + \partial_x y_1 = 0, \\ \partial_t y_2 + \partial_x y_2 = 0, \\ y_1(t,0) = y_2(t,1) + u(t) \\ y_2(t,0) = y_1(t,1), \end{cases}$$

Question

How to find a feedback control $u(t)$?

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How to find a feedback control $u(t)$?

$$\text{e.g. } u(t) = -y_2(t,1)$$

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How to find a feedback control $u(t)$ when we only measure $y_1(t,1)$?

A control $u(t) = f(y_1(t,1))$ with $f \in C^1(\mathbb{R})$ cannot work

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Idea: use an observer

Motivation

The control problem becomes

$$\begin{cases} \partial_t y_1 + \partial_x y_1 = 0, \\ \partial_t y_2 + \partial_x y_2 = 0, \\ \partial_t \hat{y}_2 + \partial_x \hat{y}_2 = 0, \end{cases}$$

with boundary conditions

$$\begin{aligned} \hat{y}_2(t,0) &= y_1(t,1) \\ y_2(t,0) &= y_1(t,1) \\ y_1(t,0) &= y_2(t,1) - \hat{y}_2(t,1). \end{aligned}$$

Proposition

This system is exponentially stable for any decay rate.

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Proposition

This system is exponentially stable for any decay rate.

... but there does not exist any diagonal quadratic Lyapunov function.

Robustness, hyperbolic systems and diffusion

Let us look at the linearized system:

$$
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\partial_t y_1 + (1+\varepsilon)\partial_x y_1 &= 0,\\
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y_1(t,0) &= y_2(t,1) - \hat{y}_2(t,1)\\
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Proposition (Bastin, Coron, A.H. 2022)

There exists arbitrarily small ε such that this system is **unstable***.*

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There exists arbitrarily small ε such that this system is ***unstable****.*

What happens if we add some viscosity?

Robustness, hyperbolic systems and diffusion

Consider the following 3×3 system

$$\begin{aligned} \partial_t y_1 + (1+\varepsilon)\partial_x y_1 - \eta\partial_{xx}^2 y_1 &= 0, \\ \partial_t y_2 + (1+\varepsilon)\partial_x y_2 - \eta\partial_{xx}^2 y_2 &= 0, \\ \partial_t \hat{y}_2 + \partial_x \hat{y}_2 &= 0, \end{aligned}$$

with boundary conditions

$$\begin{aligned} y_1(t,0) &= y_2(t,1) - \hat{y}_2(t,1) \\ y_2(t,0) &= \hat{y}_2(t,0) = y_1(t,1). \\ \partial_x y_1(t,1) &= \partial_x y_2(t,1) = 0. \end{aligned}$$

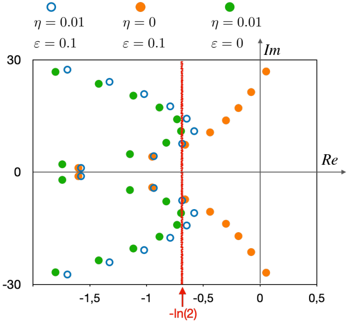
Theorem (Bastin, Coron, A.H., 2022)

For any $\delta > 0$ there exists η abitrarily small and $\varepsilon_1 > 0$ such that for any $\varepsilon \in (-\varepsilon, \varepsilon)$ the system is exponentially stable with a decay rate $\ln(2) - \delta$

Remark:

- Loss of continuity: there is a bound $\ln(2)$ on the decay rate.

Robustness, hyperbolic systems and diffusion



An illustration on the linearized Saint-Venant system

Robustness, hyperbolic systems and diffusion

Consider the following 2×2 system

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$$\begin{aligned} y(t,0) = \hat{y}(t,0) = y_1(t,1) - \hat{y}(t,1), \\ \partial_x y(t,1) = 0. \end{aligned}$$

Proposition

There exists arbitrarily small η and ε_1 such that for any $\varepsilon \in (-\varepsilon_1, \varepsilon_1)$ the system is ***unstable****.*

$\rightarrow$ Here, the addition of a small viscosity breaks the stability.

Robustness, hyperbolic systems and diffusion

- Robustness is not a given
- The effect of the viscosity is not easy to predict: in some other cases the viscosity does not improve the robustness and even breaks the stability of the linearized system.
- *The usefulness of viscosity for the robustness of boundary output feedback control of an unstable fluid flow system*, preprint, 2023, (Bastin, Coron, A.H.)

Outline of the talk

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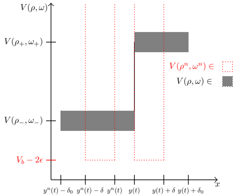

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Traffic flow instabilities: jam

What happens when you have many cars on the road with same speed and same spacing ?

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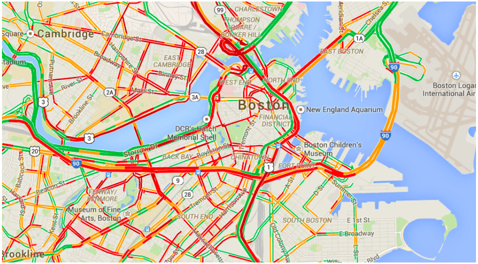
After a while you might get traffic jam.



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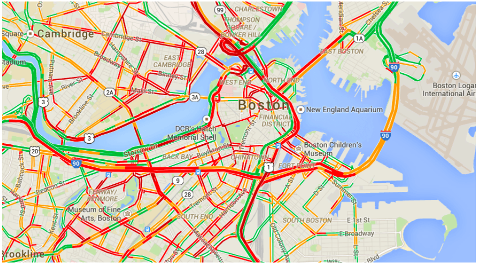

No apparent reason: no accident, no lane reduction, etc.

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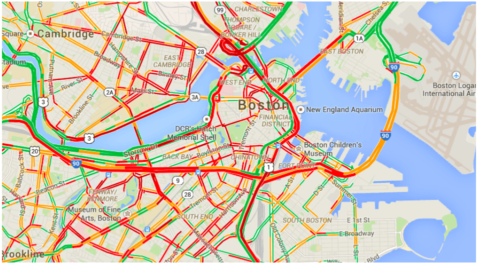

No apparent reason: no accident, no lane reduction, etc. the underlying reason is mathematical.

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Mathematical underlying reason: when density of cars is large enough steady-states are unstable.

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Can we stabilize the system using individual (autonomous) vehicles as controls. i.e. pointwise controls ?

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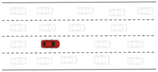

Traffic flow instabilities

Different scales

- Microscopic scale

$$\dot{x}(t) = f(t, x(t), u(t)) \quad (ODE)$$

- Macroscopic scale

$$\begin{cases} \partial_t \rho + \partial_x(\rho V(\rho)) = 0, \ (t > 0, \ x \in \mathbb{R}), \\ \dot{y}(t) = \min(u(t), V(\rho(t, y(t)+))), \end{cases}$$

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A microscopic approach

Consider a single lane ring-road of N cars

$$\begin{cases} \dot{x}_i = v_i, \\ \dot{v}_i = a\dfrac{v_{i+1} - v_i}{(x_{i+1} - x_i)^2} + b[V(x_{i+1} - x_i) - v_i], \end{cases}, \quad 1 \leq i \leq N$$

V is an equilibrium velocity, $(x_i, v_i)_{i \in \{1, \dots, N\}}$ are the variables.

Our control

$$\begin{cases} \dot{x}_{N+1} = v_{N+1} \\ \dot{v}_{N+1}(t) = u(t), \end{cases}$$

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Proposition (Cui, Seibold, Stern, Work, 2017)

Assume that $\frac{b}{2} + \frac{a}{d^2} < V'(d)$, there exists $N_1 > 0$ such that if $N > N_1$, the uncontrolled system of N cars is unstable.

The steady-state can be unstable for certain densities of cars

A microscopic approach

A single vehicle can restore the stability

Theorem (A.H., Piccoli, Truong, 2021)

Let $(\bar{v}, d)$ an admissible steady-state, if

$$u(t) = -k(v_{N+1} - \bar{v}),$$

where $k > 0$, then the system is locally asymptotically stable.

Traffic flow instabilities

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Traffic and PDEs

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ρ is the density of cars, $V(\rho)$ the speed of traffic, $y(t)$ is the location of the autonomous vehicle.

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The first equation already has a unique entropic solution of class BV. Why do we need the inequality?

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- Entropic solutions do not represent the physical solutions
- In an entropy solution, the flow would not see the AV: no creation of information at a single point $\Longrightarrow$ no macroscopic influence of a single point.
- Precisely the reason why this control can work.
- We need a new condition: the Delle-Monache Goatin flux condition.

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Theorem (Delle Monache Goatin '14, Liard Piccoli '18, Garavello Goatin Liard Piccoli '20)

There exists a unique solution $y \in W^{1,1}_{loc}(\mathbb{R}_+)$, $u \in C^0(\mathbb{R}_+, L^1)$ of bounded TV, entropic on $(-\infty, y(t))$ and $(y(t), +\infty)$.

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Unfortunately this system does not accurately represent stop-and-go waves.
Would this hold for a second-order system that does ?

A macroscopic approach: traffic and PDEs

A more involved model: GARZ equations

$$\begin{cases} \partial_t \rho + \partial_x(\rho V(\rho,\omega)) = 0, \\ \partial_t(\rho\omega) + \partial_x(\rho\omega V(\rho,\omega)) = 0 \\ \dot{y} = \min(V_b, V(\rho,\omega)). \end{cases}$$

with the Delle-Monache Goatin flux condition

$$\rho(t,y(t))(V(\rho(t,y(t)),\omega(t,y(t)) - \dot{y}) \leq \alpha \max_{\rho,\omega}(\rho(V(\rho,\omega) - u(t)))$$

Theorem (A.H., Marcellini, Liard, Piccoli, preprint)

There exists a solution in $BV(\mathbb{R};[0,\rho_{\max}]\times[\omega_{\min},\omega_{\max}])\times W^{1,1}_{loc}(\mathbb{R}_+)$, which is in addition entropic on $(-\infty,y(t))$ and $(y(t),+\infty)$.

Perspectives

Several open questions

- Can we derive a feedback to stabilize the system?
  (see Liard, Marx, Perrollaz, 2023)

- If a feedback can be derived from this system, can it be translated in the microscopic framework ?

- The coefficient α represents the proportion of space left on the road when the AV is blocking a lane. What happens in the limit $\alpha \to 0$?

CIRCLES project: a real-life application

In real-life roads several additional difficulties:

- An open system, with ramps and exits
- The model is far from being perfect (!)
- Partial measurements, loss of signal, propagating errors, imperfect actuation, etc.
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$\rightarrow$ A challenge for the CIRCLES project

CIRCLES Project

Objective: Using a few number of autonomous vehicles to stabilize the system and reduce the overall energy consumption and CO2 emissions of the traffic.

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Plan:

- Realistic simulations and controls (2020)

CIRCLES Project

Objective: Using a few number of autonomous vehicles to stabilize the system and reduce the overall energy consumption and CO2 emissions of the traffic.

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- 100 cars with the final control sent on the highway and looping on a few kilometers to represent $\sim 2\%$ of the flow (2022).

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The world's largest autonomous cars experiment in dense traffic.

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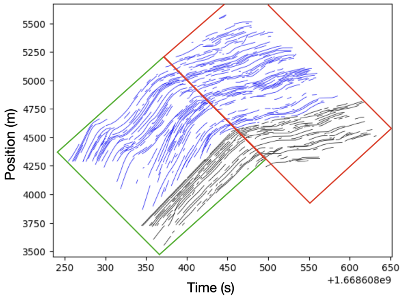
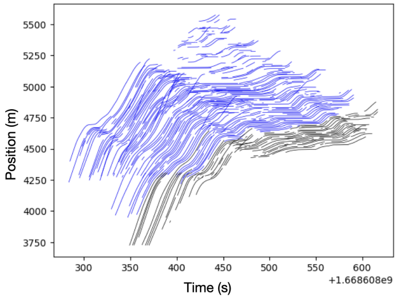
CIRCLES Project

Experimental results:
(with AlAnqary, Bhadani, Denaro, Weightman, Xiang et al.)



(a) Lane 3 (with the AV)

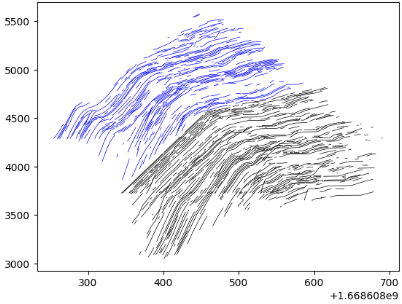


(b) Lane 1 (without the AV)

- A single AV stabilize the behavior of ∼15 cars behind it, even on the highway.
- Speed variance decrease of ∼ 50% over a wave.

CIRCLES Project

Experimental results:
(with AlAnqary, Bhadani, Denaro, Weightman, Xiang et al.)

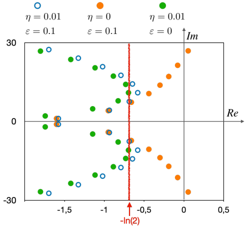


- Oscillations naturally re-appear after 15-20 cars.

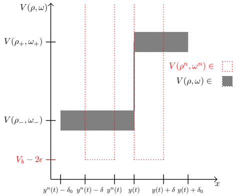
Real life experiment

… this is only the beginning.

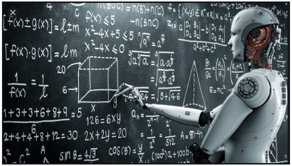
Outline of the talk



1. Robustness and hyperbolic systems



2. Control of traffic flows



3. AI and maths

Introduction

Introduction

For a year now, we have been hearing a lot about the progress of AI, particularly in one field: IA for langage.

ChatGPT

Introduction

A turning point in 2017: the Transformer

Attention is all you need
A Vaswani, N Shazeer, N Parmar… - Advances in neural …, 2017 - proceedings.neurips.cc
… to attend to **all** positions in the decoder up to and including that position. **We need** to prevent
… **We** implement this inside of scaled dot-product **attention** by masking out (setting to −∞) …

- An attention mechanism that allows it to focus on the important part of a sentence.

`Tchen tenterait-il de lever la moustiquaire ?`

AI and maths

What maths can bring to artificial intelligence (AI) vs. what AI can bring to maths ?

AI and maths

What maths can bring to artificial intelligence (AI) vs. what AI can bring to maths ? Can an AI learn mathematics in some sense ?

Two ways to see the question:

- Can it guess the solution to a mathematical problem?
- Can it prove a theorem and give the proof?

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AI and maths

Can an AI, that has no built-in math knowledge, guess the solution to a math problem? Can it learn some mathematics by examples?

AI and maths

Can an AI, that has no built-in math knowledge, guess the solution to a math problem? Can it learn some mathematics by examples? Yes, it seems.

- Guess solutions to ODE (Charton, Lample, 2019)
- Guess the controllability of a linearized system; guess a stabilizing feedback; the spectral abscissa (Charton, A.H., Lample, 2020)
- Many following works (equilibria in a graph, linear algebra, GCD, sequences etc.)

AI and maths

Two examples

- Finding a Lyapunov function
- Guessing a feedback law

AI and maths

Train an AI to guess solutions to a mathematical problem

Approach:

- Use a language model (Transformer) originally used to learn languages.
- See the problem as a translation problem between statement and solution.
- Understanding the rules hidden behind $\rightarrow$ understanding some mathematics

AI and maths

Example: finding Lyapunov functions

$$\dot{x}(t) = \begin{pmatrix} -6x_1^4(t)x_2^5(t) - 3x_1^7(t)x_3^2(t) \\ 3x_1^9(t) - 6x_1^2(t)x_2^5(t)x_3^2(t) \\ -4x_1^2(t)x_3^5(t) \end{pmatrix} \quad \rightarrow \quad V(x) = x_1^6 + 2(x_2^6 + x_3^4)$$

- An open question
- Methods exist in several cases, in particular polynomial with polynomial Lyapunov functions
- Mathematicians often rely on intuition

AI and maths

Example: finding Lyapunov functions

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- An open question
- Methods exist in several cases, in particular polynomial with polynomial Lyapunov functions
- Mathematicians often rely on intuition
- A first neural network with higher accuracy than humans (Alfarano, Charton, A.H., 2023)

AI and maths

Résultats

Type	n equations	degree	SOSTOOLS[1]	IA
polynomial	2-3	8	78%	**99.3%**
polynomial	3-6	12	16%	**95.1%**
Non-polynomial	N/A	N/A	N/A	**97.8%**
polynomial (SOSTOOLS)	2-3	6	N/A	**83.1%**

Human accuracy: XX%

[1] méthode existante.

AI and maths

Two examples

- Stability of dynamical systems
- Control theory

AI and maths

Consider the system

$$\begin{cases} \dot{E} = \beta_E F \left(1 - \dfrac{E}{K}\right) - \left(\nu_E + \delta_E\right) E, \\ \dot{M} = (1-\nu)\nu_E E - \delta_M M, \\ \dot{F} = \nu \nu_E E \dfrac{M}{M + M_s} - \delta_F F, \\ \dot{M}_s = u - \delta_s M_s, \end{cases}$$

$E(t)$ mosquitoes' eggs, $F(t)$ feconded females, $M(t)$ males, $M_s(t)$ sterilised males. u flux of released sterile mosquitoes (control). We are looking for a feedback control:

$$u = g(M + M_s, F + F_s)$$

AI and maths

$$\begin{cases} \dot{E} = \beta_E F \left(1 - \dfrac{E}{K}\right) - (\nu_E + \delta_E) E, \\ \dot{M} = (1-\nu)\nu_E E - \delta_M M, \\ \dot{F} = \nu \nu_E E \dfrac{M}{M + M_s} - \delta_F F, \\ \dot{M}_s = u - \delta_s M_s, \end{cases}$$

$$u = g(M + M_s, F + F_s)$$

Question

For ε as small as we want, is it possible to find g such that the system is stable and

$$\lim_{t \to +\infty} \|E(t), M(t), F(t)\| = 0 \ et \ \lim_{t \to +\infty} \|M_s(t)\| = \varepsilon,$$

AI and maths

$$
\begin{cases}
\dot{E} = \beta_E F \left(1 - \dfrac{E}{K}\right) - (\nu_E + \delta_E) E, \\
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\end{cases}
$$

$$
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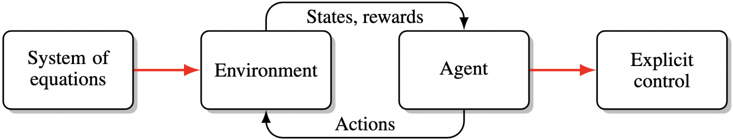
An open question

AI and maths

Principle of the approach (Agbo Bidi, Coron, A.H., Lichtlé, 2023)



AI and maths

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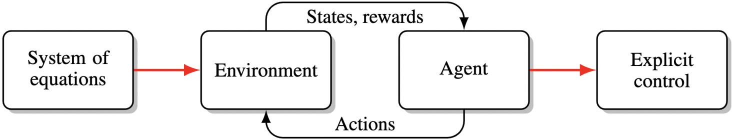
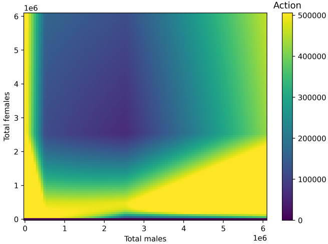


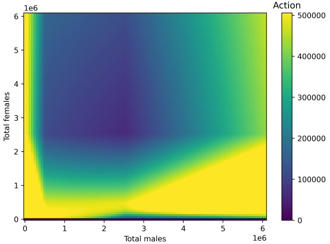
1. **Transform the equations** with a well chosen numerical scheme
2. **Train a model** based on Reinforcement Learning (RL). The AI trains by trials and errors and tries to maximize a well chosen objective.
3. **Deduce the mathematical control**, from the numerical control
4. **Check** that this is a solution to the problem.

AI and maths

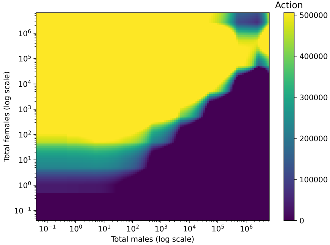


$$u = f(M + M_s, F + F_s)$$

AI and maths



$\xrightarrow{\text{N.Lichtlé}}$



$$u = f(M + M_s, F + F_s)$$

AI and maths

$$u_{\mathrm{reg}}(M+M_s,F+F_s)=\begin{cases} u_{\mathrm{reg}}^{\mathrm{left}}(M+M_s,F+F_s) & \text{if } M+M_s<M^*,\\ u_{\mathrm{reg}}^{\mathrm{right}}(M+M_s,F+F_s) & \text{otherwise,}\end{cases}$$

AI and maths

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$$u_{\text{reg}}^{\text{left}}=\begin{cases}\varepsilon & \text{if } I_1(F+F_s)>\alpha_2,\\ u_{\max}(\alpha_2-I_1) & \text{if } I_1\in(\alpha_1,\alpha_2],\\ u_{\max} & \text{otherwise, and}\end{cases}$$

$$u_{\text{reg}}^{\text{right}}=\begin{cases}\varepsilon & \text{if } I_2>\alpha_2,\\ u_{\max}(\alpha_2-I_2) & \text{if } I_2\in(\alpha_1,\alpha_2],\\ u_{\max} & \text{otherwise.}\end{cases}$$

where $I_1(x)=\frac{\log M^*}{\log(F+F_s)}$ and $I_2(x,y)=\frac{\log(M+M_s)}{\log(F+F_s)}$,

AI and maths

Final control

$$u(t) = \begin{cases} \varepsilon & \text{if } \frac{\log(M+M_s)}{\log(F+F_s)} > \alpha_2, \\ u_{\max} & \text{otherwise,} \end{cases}$$

AI and maths

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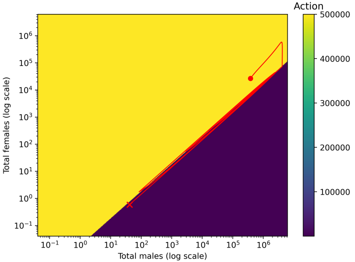

$$\varepsilon > 0$$

AI and maths

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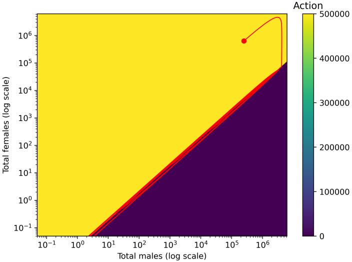

$\varepsilon = 0$

AI and maths

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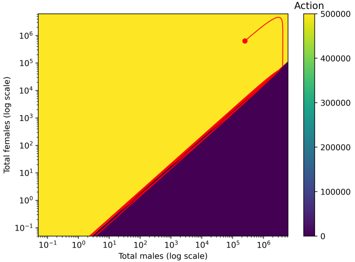

$\varepsilon = 0$

We can see a mathematical bifurcation with this "IA-augmented intuition".

AI and maths

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AI and maths

Can an AI prove a theorem and give a proof? By far the hardest question...

AI and maths

Can an AI prove a theorem and give a proof? By far the hardest question...

First approach: train a Transformer (GPT-f , Polu, Sutskever, 2020)

Question

Let $a > 0$ and $b > 0$, such that $ab = b - a$, show that

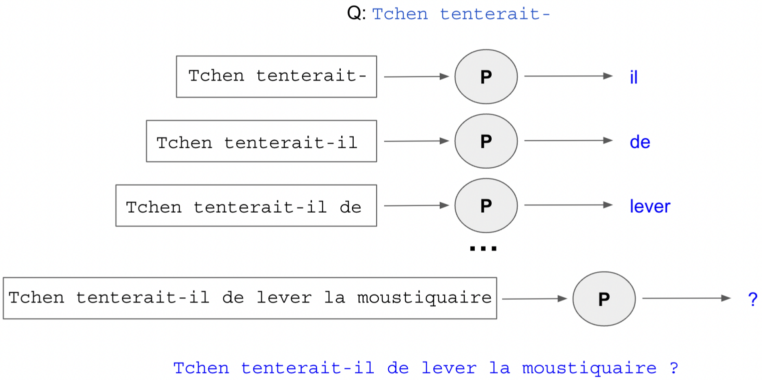
$$\frac{a}{b} + \frac{b}{a} - ab = 2$$

GPT

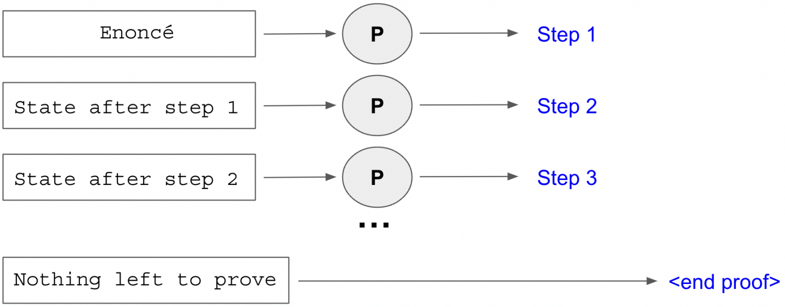
Proof

Ai and maths

2018 - GPT - an autoregressive transformer.



AI and maths



Proof: Step 1; Step 2; … ; Step n.

AI and maths

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GPT

Preuve

Procedure: train it with examples: (exercices, proofs)

- The hope is that, by showing it enough examples, the AI can learn to reason, just by predicting the next step each time.

AI and maths

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enough = diversified enough and numerous enough

AI and maths

Question

Let $a > 0$ *and* $b > 0$*, such that* $ab = b - a$*, show that*

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GPT

Proof

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enough = diversified enough and numerous enough

LeanLlama Glöckle et al. 2023 (Temperature-scaled large language models for Lean proofstep prediction)

AI and maths

Second approach: treat mathematics as a game (Lample, Lachaux, Lavril, Martinet, Hayat, Ebner, Rodriguez, Lacroix, 2022)

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AI and maths

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Deepmind (2017)

AI and maths

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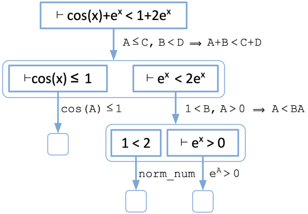
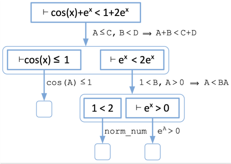


You won !

AI for maths

Main difficulties:

- two-players game vs. alone against one goal.
- In chess, when we play a move, there is still only one game. In mathematics: one statement $\longrightarrow$ many statement
- Hard in mathematics to know automatically in the middle of a proof what is the probability to succeed.
- The number of possibilities is much, much larger in mathematics



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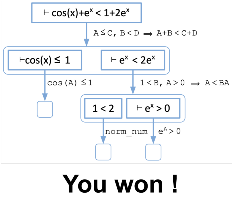


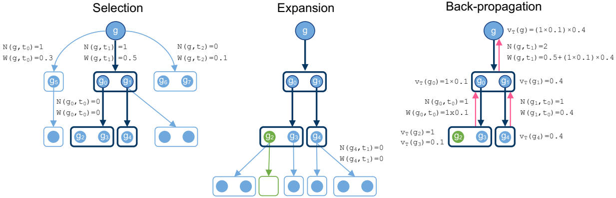
Much harder than chess

AI for maths

In practice

- Two transformers: P_θ which predicts a tactic, c_θ which predicts the probability of succeeding in proving a statement (goal, assumption, etc.).

- A clever proof search that sees the proof as a tree and combine P_θ, c_θ and an expansion of the tree.



- An online training of P_θ et c_θ depending on what has been successful.

AI for maths

Results

Undergraduate level exercices...

...30 to 60% of mid / high school exercices up to olympiads level...

...and some exercices from the International Mathematical Olympiads.

AI for maths

Results

Undergraduate level exercices...

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Exercice

Show that for any $n \in \mathbb{N}$*, 7 does not divide* $2^n + 1$*.*

Conclusion

Two subjects:

- Stabilization theory:
  - A dynamic subject using different areas of mathematics (mostly analysis)
  - Very theoretical aspects close to real-life applications
- AI for mathematics
  - A growing interest (launch of a group on automated reasoning by T. Gowers in 2022; plenary talk of K. Buzzard at ICM; T. Tao formalizing his last papers in Lean4, etc.)
  - May be a part of the future of the practice of mathematics

Conclusion

Thank you for your attention

–

Any questions ?