Switched systems with ω-regular switching sequences

Application to switched observer design

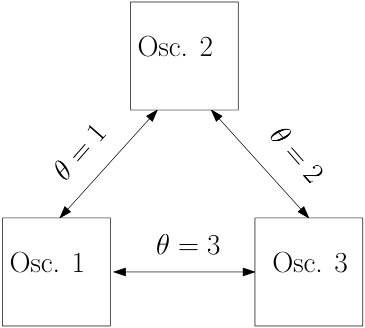
Antoine Girard, joint work with G. Aazan, L. Greco, P. Mason

Laboratoire des Signaux et Systèmes (L2S)
Université Paris-Saclay, CNRS, CentraleSupélec

Seminar d'Automatique @ Paris-Saclay
L2S, January 18, 2023

An introductory example

Network of 3 discrete-time oscillators:



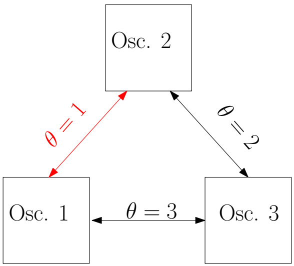
3 communication channels, only one active at any time:
$\rightarrow$ switching signal $\theta(t) \in \Sigma = \{1, 2, 3\}$.

An introductory example

Oscillator dynamics:

$$z_i(t+1) = Rz_i(t) + u_i(t),\ i = 1,2,3.$$

where $z_i(t) \in \mathbb{R}^2, u_i(t) \in \mathbb{R}^2$ and $R = \begin{pmatrix} \cos(\phi) & -\sin(\phi) \\ \sin(\phi) & \cos(\phi) \end{pmatrix}$, $\phi = \frac{\pi}{6}$.



$$u_1(t) = \begin{cases} \gamma(z_2(t) - z_1(t)), & \text{if } \theta(t) = 1 \\ 0, & \text{if } \theta(t) = 2 \\ \gamma(z_3(t) - z_1(t)), & \text{if } \theta(t) = 3 \end{cases}$$

$$u_2(t) = \begin{cases} \gamma(z_1(t) - z_2(t)), & \text{if } \theta(t) = 1 \\ \gamma(z_3(t) - z_2(t)), & \text{if } \theta(t) = 2 \\ 0, & \text{if } \theta(t) = 3 \end{cases}$$

$$u_3(t) = \begin{cases} 0, & \text{if } \theta(t) = 1 \\ \gamma(z_2(t) - z_3(t)), & \text{if } \theta(t) = 2 \\ \gamma(z_1(t) - z_3(t)), & \text{if } \theta(t) = 3 \end{cases}$$

An introductory example

Error dynamics:

$$x(t+1) = A_{\theta(t)} x(t)$$

where $x(t) = \begin{pmatrix} z_2(t) - z_1(t) \\ z_3(t) - z_2(t) \end{pmatrix}$ and

$$A_1 = \begin{pmatrix} R-2\gamma I_2 & 0 \\ \gamma I_2 & R \end{pmatrix}, A_2 = \begin{pmatrix} R & \gamma I_2 \\ 0 & R-2\gamma I_2 \end{pmatrix}, A_3 = \begin{pmatrix} R-\gamma I_2 & -\gamma I_2 \\ -\gamma I_2 & R-\gamma I_2 \end{pmatrix}.$$

To synchronize the oscillators, we impose a fairness constraint that θ cannot keep activating the same communication channel.

$$\forall t \in \mathbb{N}, \exists t' \geq t,\ \theta(t') \neq \theta(t).$$

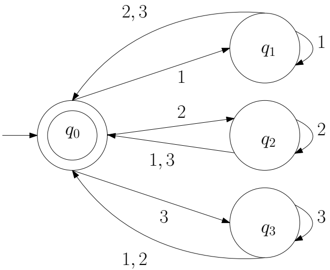
$\rightarrow$ This is an example of an ω-regular language[1].

[1] Baier & Katoen, *Principles of model checking*. 2008

ω-regular languages

ω-regular languages are those that are characterized by Büchi automata.

Example:



$$\mathcal{B} = (Q, \Sigma, q_0, \delta, F),$$
$$Q = \{q_0, q_1, q_2, q_3\},$$
$$\Sigma = \{1, 2, 3\},$$
$$F = \{q_0\}.$$

- A run $q : \mathbb{N} \to Q$ with $q(0) = q_0$ and associated with a sequence $\theta : \mathbb{N} \to \Sigma$ is said to be accepting if $q(t) \in F$ for infinitely many $t \in \mathbb{N}$.
- The language of $\mathcal{B}$, denoted by $Lang(\mathcal{B})$, consists of all sequences having an accepting run in $\mathcal{B}$.

ω-regular languages

Useful to model many natural switching constraints:

- Shuffled switching signals:

  Each mode is activated an infinite number of times.

- Persistent connectivity:

  At all time, the union of future communication graphs is connected.

- Linear temporal logic (LTL):

  LTL formulas are commonly used to specify protocols in distributed communication/computation architectures.

In general, ω-regular languages cannot be captured by dwell-time constraints or graph-constrained switching signals.

Talk outline

1. Stability analysis of systems with ω-regular switching sequences:

   Aazan, Girard, Mason, & Greco, *Stability of discrete-time switched linear systems with ω-regular switching sequences*. HSCC 2022.

   Aazan, Girard, Mason, & Greco, *A joint spectral radius for ω-regular language driven switched linear systems*. In Hybrid and Networked Dynamical Systems - Modeling, Analysis and Control, to appear.

2. Application to switched observer design:

   Aazan, Girard, Greco, & Mason, *An automata theoretic approach to observer design for switched linear systems*. Submitted.

Switched systems with ω-regular switching sequences

- Consider a discrete-time switched linear system:

$$x(t+1) = A_{\theta(t)} x(t)$$

where $\theta(t) \in \Sigma = \{1, \cdots, m\}$ is a discrete switching variable, $x(t) \in \mathbb{R}^n$ is the continuous state vector and $\mathcal{A} = \{A_1, \cdots, A_m\}$ is a finite set of matrices.

- Study the stability under ω-regular switching signals generated by a Büchi automaton $\mathcal{B}$.

- Our goal: sufficient and necessary conditions for stability of $(\mathcal{A}, \mathcal{B})$.

Stability notions

Consider a switched system $(\mathcal{A}, \mathcal{B})$ defined by a set of matrices $\mathcal{A}$ and a Büchi automaton $\mathcal{B}$.

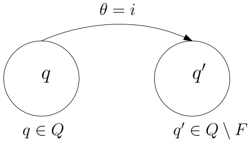
Definition (Global asymptotic stability)

$(\mathcal{A}, \mathcal{B})$ is globally asymptotically stable (GAS) if there exists $\alpha \geq 1$ such that for all $\theta \in Lang(\mathcal{B})$ and for all $x_0 \in \mathbb{R}^n$, we have:

- stability: $\|\mathbf{x}(t, x_0, \theta)\| \leq \alpha \|x_0\|, \forall t \in \mathbb{N}$;
- global attractivity: $\lim_{t \to \infty} \|\mathbf{x}(t, x_0, \theta)\| = 0$.

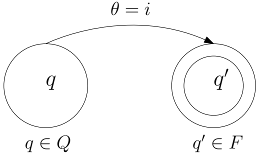
A Lyapunov approach

Consider a candidate Lyapunov function $V : Q \times \mathbb{R}^n \to \mathbb{R}_0^+$



$V(q,x) \geq V(q', A_i x)$

A transition toward a non-accepting state.



$V(q,x) > V(q', A_i x)$

A transition toward an accepting state.

The function $V(q(t), x(t))$ is

- always non-increasing
  $\implies$ stability
- strictly decreasing when an accepting state is visited
  $\implies$ global attractivity

Necessary and sufficient conditions

Theorem

If there exist a function $V : Q \times \mathbb{R}^n \to \mathbb{R}_0^+$, *scalars* $\alpha_1, \alpha_2 > 0$ *and* $\lambda \in (0,1)$ *such that for all* $x \in \mathbb{R}^n$:

$$
\begin{aligned}
&\alpha_1 \|x\| \le V(q,x) \le \alpha_2 \|x\|, && q \in Q \\
&V(q', A_i x) \le V(q,x), && q \in Q, i \in \Sigma, q' \in \delta(q,i) \setminus F \\
&V(q', A_i x) \le \lambda V(q,x), && q \in Q, i \in \Sigma, q' \in \delta(q,i) \cap F
\end{aligned}
$$

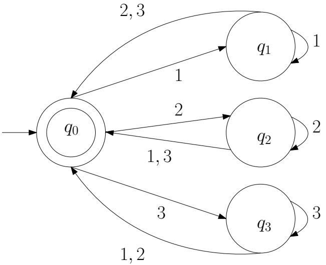
then $(\mathcal{A}, \mathcal{B})$ *is GAS.*

Conversely, if all matrices in $\mathcal{A}$ *are invertible and* $(\mathcal{A}, \mathcal{B})$ *is GAS, then there exists such a Lyapunov function.*

In some cases, V can be taken quadratic of the form $V(q,x) = \sqrt{x^\top P_q x}$, $P_q \in \mathbb{R}^{n \times n}$, then the stability of $(\mathcal{A}, \mathcal{B})$ can be verified by solving LMIs.

Numerical example - oscillator network

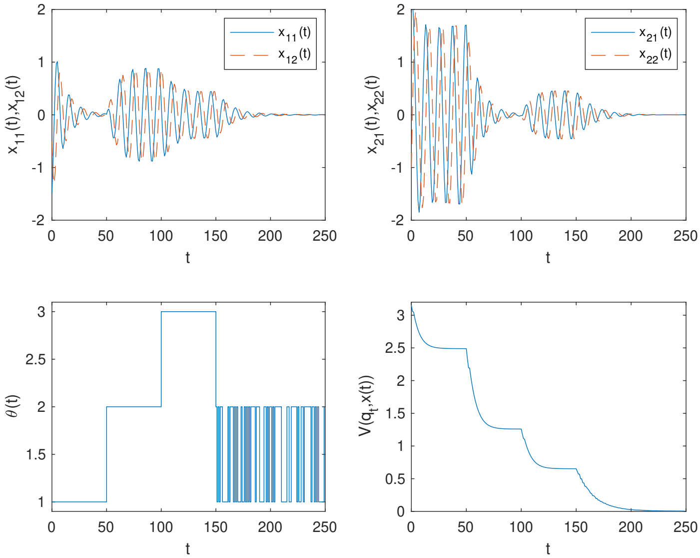
Consider our introductary example and let the candidate Lyapunov function V be quadratic of the form $V(q,x) = \sqrt{x^\top P_q x}$.



$$\begin{cases} I_4 \leq P_q, & q \in Q \\ A_i^\top P_{q'} A_i \leq P_q, & q \in Q, i \in \Sigma, q' = \delta(q,i) \neq q_0 \\ A_i^\top P_{q_0} A_i \leq \lambda^2 P_q & q \in Q, i \in \Sigma, \delta(q,i) = q_0. \end{cases}$$

Solving these 16 LMIs numerically, we get a solution for $\lambda = 0.96$.

Numerical example - oscillator network



Characterization of the convergence rate

Consider an ω-regular switching signal θ belonging to the language of a Büchi automaton $\mathcal{B} = (Q, \Sigma, q_0, \delta, F)$.

Then, we define:

- The sequence of return instants:

$$\tau_0^\theta = 0,\ \tau_{k+1}^\theta = \min\{t > \tau_k^\theta \mid q_t \in F\}$$

- The shuffling index:

$$\kappa^\theta(t) = \max\{k \in \mathbb{N} \mid \tau_k^\theta \leq t\}$$

- The accepting rate:

$$\gamma^\theta = \liminf_{t \to \infty} \frac{\kappa^\theta(t)}{t}$$

Characterization of the convergence rate

Theorem

If there exist a function $V : Q \times \mathbb{R}^n \to \mathbb{R}_0^+$, *scalars* $\alpha_1, \alpha_2, \rho > 0$, *and* $\lambda \in (0,1)$ *such that for all* $x \in \mathbb{R}^n$, *the following hold:*

$$\begin{aligned} &\alpha_1 \|x\| \leq V(q,x) \leq \alpha_2 \|x\|, && q \in Q \\ &V(q', A_i x) \leq \rho V(q,x), && q \in Q, i \in \Sigma, q' \in \delta(q,i) \setminus F \\ &V(q', A_i x) \leq \rho \lambda V(q,x), && q \in Q, i \in \Sigma, q' \in \delta(q,i) \cap F \end{aligned}$$

then, there exists $C \geq 1$ *such that for all* $x_0 \in \mathbb{R}^n$, *and for all* $\theta \in Lang(\mathcal{B})$:

$$\forall t \in \mathbb{N},\ \|\mathbf{x}(t, x_0, \theta)\| \leq C \rho^t \lambda^{\kappa^\theta(t)} \|x_0\|. \tag{1}$$

Conversely, if all matrices in $\mathcal{A}$ *are invertible and* (1) *holds, then there exists such a Lyapunov function.*

A partial stability result

Even if $\rho > 1$, the system can be stable provided the accepting states are visited sufficiently often:

Corollary

Consider a function $V : Q \times \mathbb{R}^n \to \mathbb{R}_0^+$ *as in the previous theorem. Let* $\theta \in Lang(\mathcal{B})$ *such that* $\gamma^\theta > -\frac{\ln(\rho)}{\ln(\lambda)}$, *then*

$$\lim_{t\to\infty} \|\mathbf{x}(t, x_0, \theta)\| = 0, \ \forall x_0 \in \mathbb{R}^n.$$

Talk outline

1. Stability analysis of systems with ω-regular switching sequences:

   Aazan, Girard, Mason, & Greco, *Stability of discrete-time switched linear systems with ω-regular switching sequences*. HSCC 2022.

   Aazan, Girard, Mason, & Greco, *A joint spectral radius for ω-regular language driven switched linear systems*. In Hybrid and Networked Dynamical Systems - Modeling, Analysis and Control, to appear.

2. Application to switched observer design:

   Aazan, Girard, Greco, & Mason, *An automata theoretic approach to observer design for switched linear systems*. Submitted.

Switched systems with unobservable modes

- Consider a discrete-time switched linear system:

$$
\begin{aligned}
x(t+1) &= A_{\theta(t)}x(t) \\
y(t) &= C_{\theta(t)}x(t)
\end{aligned}
$$

  where $\theta(t) \in \Sigma = \{1, \cdots, m\}$ is a discrete switching variable, $x(t) \in \mathbb{R}^n$ is the continuous state vector, $y(t) \in \mathbb{R}^p$ is the output, $\mathcal{A} = \{A_1, \cdots, A_m\}$ and $\mathcal{C} = \{C_1, \cdots, C_m\}$ are finite sets of matrices.

- We assume the system is unobservable for arbitrary switching (e.g. if some pairs (A_i, C_i) are unobservable).

- Our goal: identify a large set of "observable" switching signals and propose an approach for asymptotic observer design.

Observability of discrete-time switched systems

Definition (Reconstructibility)

The switched system is reconstructible, if there exist $k \in \mathbb{N}$ and θ such that the knowledge of $y(0), \dots, y(k)$ is sufficient to determine $x(k)$. $\theta(0), \dots, \theta(k)$ is called a reconstructible sequence.

Theorem (Sun & Ge 2005)

A sequence of modes $i_1, \dots, i_j \in \Sigma$ is reconstructible if and only if

$$\ker\Big(\Omega(i_1, \dots, i_j)\Big) \subseteq \ker(A_{i_1} \cdots A_{i_j}).$$

where $\Omega(i_1, \dots, i_j) = \begin{bmatrix} C_{i_1}^\top & A_{i_1}^\top C_{i_2}^\top & \cdots & A_{i_1}^\top \cdots A_{i_{j-1}}^\top C_{i_j}^\top \end{bmatrix}^\top.$

Observability of discrete-time switched systems

Claim

To be able to "robustly" estimate the state of the system, the switching signal needs to contain an infinite number of reconstructible sequences.

- For $k \in \mathbb{N}$, let $\mathcal{O}^{[k]}$ denote the set of "minimal" reconstructible sequences of length at most k.
- Let us consider $(\Sigma^*\mathcal{O}^{[k]})^\omega$, the set of switching signals containing an infinite number of sequences in $\mathcal{O}^{[k]}$.
- $(\Sigma^*\mathcal{O}^{[k]})^\omega$ is an ω-regular language, we denote by $\mathcal{B}_k$ the associated Büchi automaton.

Example

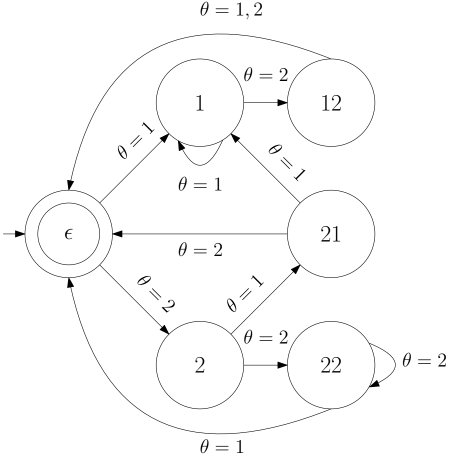
$$A_1 = I_3$$

$$A_2 = 1.5 \times \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$$C_1 = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}$$

$$C_2 = \begin{pmatrix} 0 & 1 & 1 \end{pmatrix}$$

$$\mathcal{O}^{[3]} = \{121, 122, 212, 221\}$$



$$Lang(\mathcal{B}_k) = (\Sigma^* \mathcal{O}^{[3]})^\omega$$

Observer structure

Let $\mathcal{B}_k = (Q, \Sigma, \delta, q_0, F)$, we consider a switched observer with an internal discrete state $q(t) \in Q$

$$q(t+1) = \delta(q(t), \theta(t)),\ q(0) = q_0,$$
$$\hat{x}(t+1) = A_{\theta(t)}\hat{x}(t) + L_{(q(t),\theta(t))}\left(y(t) - C_{\theta(t)}\hat{x}(t)\right).$$

The dynamics of q is given by the transition function δ of $\mathcal{B}_k$.

Consider the estimation error $e(t) = x(t) - \hat{x}(t)$, then

$$e(t+1) = (A_{\theta(t)} - L_{(q(t),\theta(t))}C_{\theta(t)})e(t).$$

We want to ensure stability of the error dynamics for all $\theta \in Lang(\mathcal{B}_k)$.

Lyapunov conditions for observer design

Proposition

Let us assume that there exist $V : Q \times \mathbb{R}^n \to \mathbb{R}_0^+$, $\alpha_1, \alpha_2, \rho > 0$, *and* $\lambda \in (0,1)$, *such that for all* $e \in \mathbb{R}^n$:

$$\alpha_1 \|e\| \leq V(q,e) \leq \alpha_2 \|e\|, \qquad q \in Q \quad (2)$$

$$V\left(q', (A_i - L_{(q,i)} C_i) e\right) \leq \rho V(q,e), \qquad q \in Q, i \in \Sigma, q' = \delta(q,i) \notin F \quad (3)$$

$$V\left(q', (A_i - L_{(q,i)} C_i) e\right) \leq \rho \lambda V(q,e), \qquad q \in Q, i \in \Sigma, q' = \delta(q,i) \in F \quad (4)$$

Then, there exists $C \geq 1$ *such that for all* $e_0 \in \mathbb{R}^n$, $\theta \in Lang(\mathcal{B}_k)$:

$$\forall t \in \mathbb{N}, \; \|\mathbf{e}(t, e_0, \theta)\| \leq C \rho^t \lambda^{\kappa^{\theta, \mathcal{B}_k}(t)} \|e_0\|.$$

Moreover, whenever the accepting rate $\gamma^\theta > -\frac{\ln(\rho)}{\ln(\lambda)}$, *we have*

$$\lim_{t \to \infty} \|\mathbf{e}(t, e_0, \theta)\| = 0.$$

Observer gain design

Proposition

Let $\rho > 0$ *and* $\lambda \in (0,1)$. *Let us assume that there exist* $P_q \in \mathbb{R}^{n \times n}$, $Y_{(q,i)} \in \mathbb{R}^{n \times p}$, *for* $q \in Q$, $i \in \Sigma$, *such that the following LMIs hold:*

$$
\begin{aligned}
& P_q > 0 && q \in Q, && (5)\\
& \begin{pmatrix} P_{q'} & P_{q'}A_i - Y_{(q,i)}C_i \\ \star & \rho^2 P_q \end{pmatrix} \geq 0 && q \in Q, i \in \Sigma, q' = \delta(q,i) \notin F && (6)\\
& \begin{pmatrix} P_{q'} & P_{q'}A_i - Y_{(q,i)}C_i \\ \star & \rho^2 \lambda^2 P_q \end{pmatrix} \geq 0 && q \in Q, i \in \Sigma, q' = \delta(q,i) \in F && (7)
\end{aligned}
$$

Then, the function $V(w,e) = \sqrt{e^\top P_w e}$ *satisfies inequalities* (2),(3) *and* (4) *with observer gains*

$$L_{(q,i)} = P_{\delta(q,i)}^{-1} Y_{(q,i)}, \; q \in Q, \; i \in \Sigma.$$

A converse result

Let $\rho_e(\mathcal{A})$ denote the ellipsoid norm approximation of the joint spectral radius of $\mathcal{A}$:

$$\rho_e(\mathcal{A}) = \inf\left\{\rho \geq 0 \;\middle|\; \begin{array}{c} \exists M > 0,\ M^\top = M, \\ \forall A \in \mathcal{A},\ A^\top M A \leq \rho^2 M \end{array}\right\}.$$

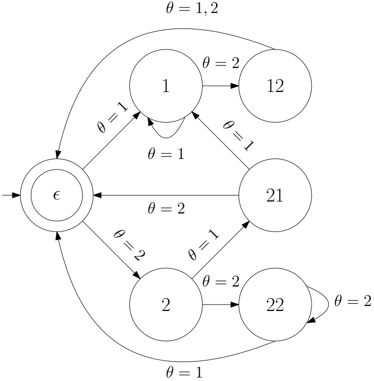
Theorem

Let us assume all matrices in $\mathcal{A}$ are invertible. Then, LMIs (5), (6), (7) have a feasible solution for all $\rho > \rho_e(\mathcal{A})$ and $\lambda \in (0,1)$.

- The proof provides explicit expression of the solution of LMIs
- Near universal approach to observer design for switched systems
- The observer is robust to noise in the ISS sense

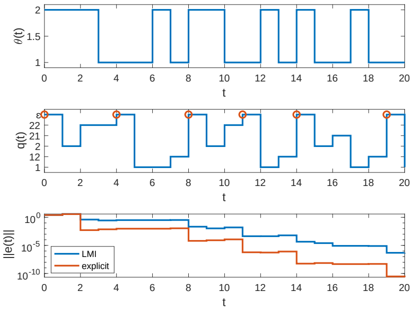
Example

$$A_1 = I_3$$
$$A_2 = 1.5 \times \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$
$$C_1 = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}$$
$$C_2 = \begin{pmatrix} 0 & 1 & 1 \end{pmatrix}$$
$$\mathcal{O}^{[3]} = \{121, 122, 212, 221\}$$

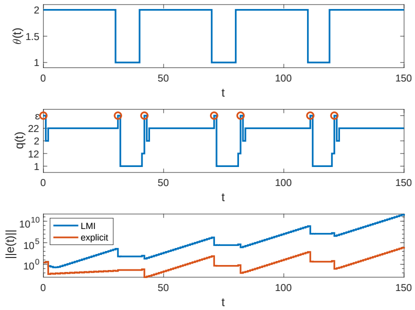


LMIs (5), (6), (7) solved for $\rho = 1.5$ and $\lambda = 0.1$:
$\implies$ observer converges for switching signals with $\gamma^\theta > 0.18$.

Example

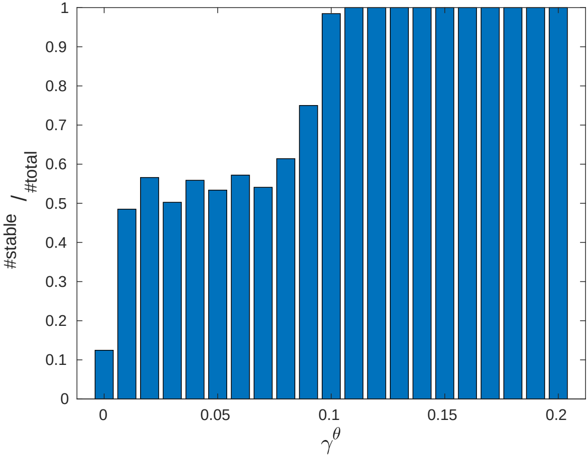


$$\gamma^\theta = 0.25 > 0.18$$



$$\gamma^\theta = 0.05 < 0.18$$

Example

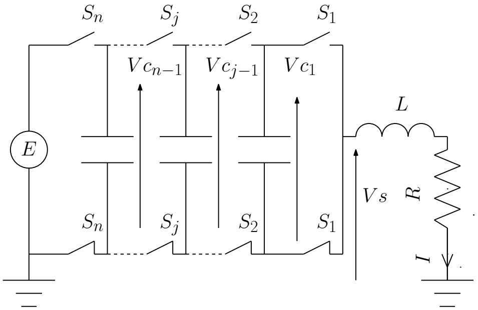


Statistics over 10 000 simulations

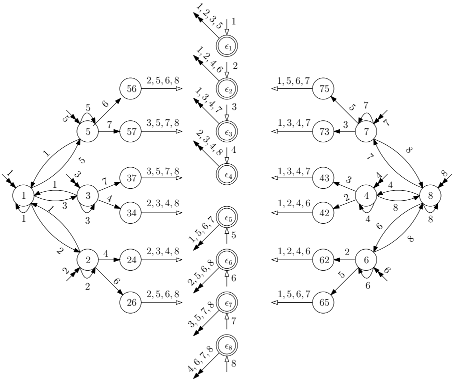
Case study - multicellular converter

3 commutation cells:

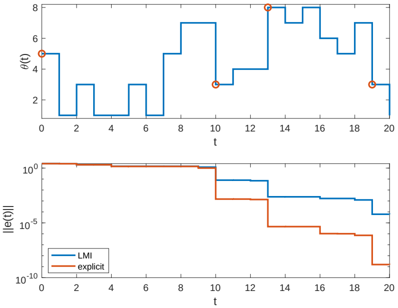
- 3 states: V_{c_1}, V_{c_2}, I
- 1 output: I
- 8 modes + constraints



- The dynamics in each mode is unobservable
  $\implies$ unobservable for arbitrary switching
- For $k = 3$, the set $\mathcal{O}^{[k]}$ contains 48 minimal reconstructible sequences
- LMIs (5), (6), (7) solved for $\rho = 1$ and $\lambda = 0.1$.



Büchi automaton $\mathcal{B}_k$



Simulation result

Conclusion and perspectives

Systems under ω-regular switching signals:

- Stability analysis using Lyapunov functions and automata-theoretic techniques.
- Application to observer design for switched systems.

Perspectives and future work:

- Computation of non-quadratic Lyapunov functions (e.g. path-complete Lyapunov functions).
- Language theoretic characterization of reconstructible sequences without bound on the length.
- Controller design for switched systems (not an easy extension).