Analysis and Control of Multi-timescale Modular Directed Heterogeneous Networks

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Overview

Generalities on modular networks

We study the collective behavior of heterogeneous nonlinear systems, interconnected over generic **directed** graphs, in the scenario that, due to the nature of their interconnections, the agents self-organize in modules.

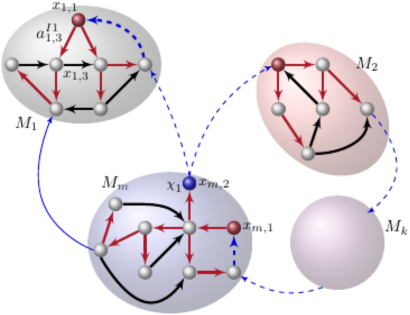

Figure: Schematic representation of a directed modular network composed of m modules M_k, each constituting a sub-network

Generalities on modular networks

Such networks know various applications such as **power networks** or **opinion dynamics**.

- The agents of the same module tend to find a **local consensus** very quickly before **synchronizing** with the rest of the network at a slower pace Romeres, Dörfler, and Bullo 2013; Varma, Morarescu, and Hayel 2018.

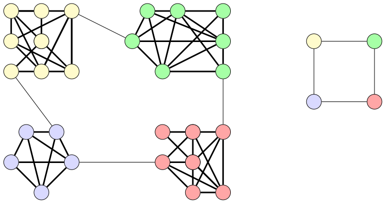

Figure: Original network and its corresponding **connected** reduced order network

Problem formulation

Problem formulation

We are interested in investigating sufficient conditions to guarantee **global asymptotic stability of the origin** for modular networks, but also in devising **control design strategies to stabilize the origin**. More precisely, we consider the nonlinear dynamics,

$$\dot{x}_{k,i} = f_{k,i}(x_{k,i}) + u_{k,i}, \quad k \leq m, \; i \leq N_k, \quad x \in \mathbb{R}^{n_x}. \tag{1}$$

Assumption (regularity and passivity)

For each pair (k,i), the function $f_{k,i}$ in (1) is continuously differentiable and admits the origin as a unique equilibrium. In addition, all the units (1) are semi-passive with respect to the input $u_{k,i}$ and the output $x_{k,i}$. That is, there exist positive definite and radially unbounded storage functions $V_{k,i}$, positive constants $\rho_{k,i}$, continuous functions $H_{k,i}$, and non-negative continuous functions $\psi_{k,i}$ such that

$$\dot{V}_{k,i}(x_{k,i}) \leq x_{k,i} u_{k,i} - H_{k,i}(x_{k,i})$$

and $H_{k,i}(x_{k,i}) \geq \psi_{k,i}(|x_{k,i}|)$ for all $|x_{k,i}| \geq \rho_{k,i}$.

Problem formulation

Assume the network's topology invariant. In compact form, for $L := L^I + L^E$, the input $u := -\sigma^I[L^I \otimes I_{n_x}]x - \sigma^E[L^E \otimes I_{n_x}]x$ and,

$$\dot{x} = f(x) - \sigma^I[L^I \otimes I_{n_x}]x - \sigma^E[L^E \otimes I_{n_x}]x, \tag{2}$$

where $L^I = \mathrm{blockdiag}\left[\, L_1^I \;\; L_2^I \;\; \cdots \;\; L_m^I \,\right], \quad L_k^I \in \mathbb{R}^{N_k \times N_k},\; k \leq m$ and $L^E = L - L^I$.

Assumption (topology)

Each module M_k individually forms a strongly-connected sub-network; its topology is captured by the Laplacian L_k^I. Furthermore, both the overall network and the network of modules, contain a spanning tree. In addition, the interconnection strengths satisfy $\sigma^E|L^E| < \sigma^I|L^I|$, where $|\cdot|$ denotes the induced L_2 norm.

Time-scale separation

Consider the kth module containing N_k nodes, and consisting in a connected (sub)network with Laplacian L_k^I. The latter admits the Jordan's decomposition

$$L_k^I = V_k \begin{bmatrix} 0 & 0 \\ 0 & \Lambda_k^I \end{bmatrix} V_k^- \qquad V_k = [1_{N_k} \; Q_k]. \tag{3}$$

Then, we introduce the new variables $\zeta_k \in \mathbb{R}^{n_x}$ and $\xi_k \in \mathbb{R}^{n_x N_k - 1}$

$$\begin{bmatrix} \zeta_k \\ \xi_k \end{bmatrix} = \begin{bmatrix} [v_{\ell k}^\top \otimes I_{n_x}] \\ [Q_k^\dagger \otimes I_{n_x}] \end{bmatrix} \bar{x}_k,$$

where $\bar{x}_k := [x_{k,1}^\top \; x_{k,2}^\top \; \ldots \; x_{k,N_k}^\top]^\top$ denotes the vector of states corresponding to all the nodes in the kth module.

The state variable ζ_k may be regarded as the weighted average of the kth module's nodes' states while ξ_k is a projection of the synchronization errors, $e_k \in \mathbb{R}^{n_x N_k}$, that is,

$$e_k := \bar{x}_k - [1_{N_k} \otimes I_{n_x}]\zeta_k = [Q_k \otimes I_{n_x}]\xi_k. \tag{4}$$

Hence, $\xi_k = 0$ if and only if the nodes in the kth cluster synchronize with the dynamics of the corresponding averaged system.
Let $\zeta \in \mathbb{R}^m$, $\zeta := [\zeta_1^\top \ \cdots \ \zeta_m^\top]^\top$ and $\xi \in \mathbb{R}^{N_{k-1}m}$, $\xi := [\xi_1^\top \ \cdots \ \xi_m^\top]^\top$, for the whole network we have

$$\begin{bmatrix} \zeta \\ \xi \end{bmatrix} = \begin{bmatrix} P^\dagger \\ Q^\dagger \end{bmatrix} x, \tag{5}$$

where P and Q are defined as $P^\dagger := \text{blockdiag}_{k\leq m}\{v_{\ell k}\} \otimes I_{n_x}$, $Q^\dagger := \text{blockdiag}_{k \leq m}\{Q_k^\dagger\} \otimes I_{n_x}$

Time-scale separation: intra-modular dynamics

Hence, differentiating on both sides of (5) we obtain

$$\dot{\zeta} = f_1(\zeta, \xi)$$
$$\dot{\xi} = -\sigma^I(\Lambda^I \otimes I_{n_x})\xi + f_2(\zeta, \xi),$$

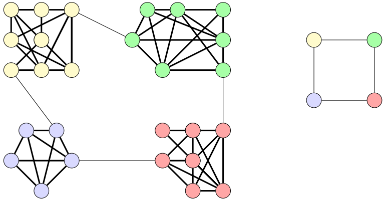

Figure: Original network and its corresponding **connected** reduced order network

where

- $-\Lambda^I := \text{blockdiag}_{k \leq m}\{\Lambda^I_k\}$ is Hurwitz,
- $f_1(\zeta, 0) = P^\dagger f\left(\bar{P}\zeta\right) - \sigma^E P^\dagger[L^E \otimes I_{n_x}]\bar{P}\zeta$ – slow "average" dynamics

Time-scale separation

Let $\varepsilon_E := 1/\sigma^E$ and the ratio of influence between and within modules, $\mu := \frac{\sigma^E}{\sigma^I \underline{\Lambda}^I}$, where $\underline{\Lambda}^I$ is the lowest eigenvalue of $\bar{\Lambda}^I$. Then, we introduce the second parameter $\varepsilon_I := \mu\varepsilon_E = \frac{1}{\sigma^I \underline{\Lambda}^I}$, so the system may be written in the form,

$$\dot{x}_e = f_e(x_e, \eta, \xi) \tag{7a}$$

$$\epsilon_E \dot{\eta} = -\bar{\Lambda}^E \eta + \epsilon_E f_\eta(x_e, \eta, \xi) \tag{7b}$$

$$\epsilon_I \dot{\xi} = -\bar{\Lambda}^I \xi + \epsilon_I f_\xi(x_e, \eta, \xi). \tag{7c}$$

The system (7) is in standard singular perturbation form Khalil 2002, albeit with three time-scales.

Main result

Main result

Theorem

Consider the networked system (2) under Assumptions 1 and 2. Assume that for any $R > 0$*, there exists a positive-definite decrescent, once continuously differentiable, function* $V_e : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ *and positive constants* q_1 *and* c_1*, such that*

$$\frac{\partial V_e}{\partial x_e} f_e(x_e, 0, 0) \leq -q_1 |x_e|^2 \tag{8}$$

$$\left| \frac{\partial V_e}{\partial x_e} \right| \leq c_1 |x_e|, \tag{9}$$

for all $x \in B_R$*. Then, there exist* $\sigma^{E^*} > 0$ *and* $\sigma^{I^*} > 0$ *such that, for all* $\sigma^I > \sigma^{I^*}$ *and* $\sigma^E > \sigma^{E^*}$*, the origin for (2) is globally asymptotically stable.*

Lemma (corollary of Theorem 11.3 in Khalil 2002- I)

Consider the nonlinear autonomous singularly-perturbed system,

$$\dot{x} = f(x, z) \tag{10a}$$

$$\epsilon \dot{z} = Az + \epsilon g(x, z), \tag{10b}$$

where $x \in \mathbb{R}^{n_x}$, $z \in \mathbb{R}^{n_z}$ *and* $A \in \mathbb{R}^{n_z \times n_z}$ *Hurwitz. Assume that the equilibrium* $(x, z) = (0, 0)$ *is an isolated equilibrium point and, for any* $R > 0$, f *and* g *are Lipschitz for all* $(x, z) \in B_R$, *with* $B_R := \{(x, z) \in \mathbb{R}^{n_x \times n_z} : |x|^2 + |z|^2 < R^2\}$, *with a Lipschitz constant* $L(R)$.
Let $L_1(R) > 0$ *and* $L_2(R) > 0$ *Lipschitz constants satisfying on* B_R:

$$|g(x, z) - g(0, z)| \leq L_1(R)|x|, \quad |g(0, z)| \leq L_2(R)|z|. \tag{11}$$

Sketch of the proof

Lemma (corollary of Theorem 11.3 in Khalil 2002- II)

In addition, assume that for each $R > 0$ *there exist positive definite decrescent functions* V *and* $W : B_R \to \mathbb{R}_{\geq 0}$*, positive constants* α_1*,* α_2*,* β*, as well as positive-definite functions* $\phi_1 : B_R \to \mathbb{R}$ *and* $\phi_2 : B_R \to \mathbb{R}$*, given by* $\phi_1(x) = |x|$*,* $\phi_2(z) = |z|$*, such that, for all* $(x, z) \in B_R$*,*

$$\frac{\partial V}{\partial x} f(x, 0) \leq -\alpha_1 \phi_1(x)^2, \tag{12}$$

$$\frac{\partial W}{\partial z} Az \leq -\alpha_2 \phi_2(z)^2, \tag{13}$$

$$\frac{\partial V}{\partial x}[f(x, z) - f(x, 0)] \leq \beta \phi_1(x) \phi_2(z). \tag{14}$$

Then, for all $\epsilon < \epsilon^* := \frac{\alpha_1 \alpha_2}{\alpha_1 L_2(R) + \beta L_1(R)} > 0$*, the origin of* (10) *is asymptotically stable and attractive to all trajectories that are contained in* B_R*.*

Sketch of the proof

The proof consists in applying Lemma 2 twice consecutively. One first time to show global asymptotic stability of the origin for the inter-modular dynamics on the synchronization manifold,

$$\dot{x}_e = f_e(x_e, \eta, 0) \tag{15a}$$

$$\dot{\eta} = -\sigma_E \bar{\Lambda}^E \eta + f_\eta(x_e, \eta, 0), \tag{15b}$$

and a second time for the intra-modular dynamics

$$\dot{y} = f_y(y, \xi), \tag{16a}$$

$$\dot{\xi} = -\sigma^I \bar{\Lambda}^I \xi + g_\xi(y, \xi), \tag{16b}$$

where $y = [x_e^\top \ \eta^\top]^\top$,

$$g_\xi(y, \xi) = F_\xi([W^\dagger \otimes I_n]y, \xi),$$

$$f_y(y, \xi) := \begin{bmatrix} f_e(x_e, \eta, \xi) \\ -\sigma^E \bar{\Lambda}^E \eta + f_\eta(x_e, \eta, \xi), \end{bmatrix}.$$

Application to network stabilization

Stabilization via control nodes added to modules

Corollary

Consider the networked system

$$\dot{x} = F(x) - \sigma^I[L^I \otimes I_n]x - \sigma^E[L^E \otimes I_n]x + v(x, x_c), \quad (17)$$

under Assumptions 2 and 1, and the dynamic control extensions

$$\dot{x}_{k,N_k+j} = f_{k,N_k+j}(x_{k,N_k+j}) - \sigma^I \sum_{i=1}^{N_k} a^I_{k,j,i}(x_{k,N_k+j} - x_{k,i}). \quad (18)$$

Assume, in addition, that the origin for $\dot{x}_e = \sum_{k=1}^{m} \sum_{i=1}^{N_k+N'_k} w_{\ell k} v_{\ell k i} f_{k,i}(x_e)$, *is globally asymptotically stable and there exists a continuously differentiable Lyapunov function* $V_e : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ *satisfying (8)-(9). Then, there exist* $\sigma^{E^*} > 0$ *and* $\sigma^{I^*} > 0$ *such that, for all* $\sigma^I > \sigma^{I^*}$ *and* $\sigma^E > \sigma^{E^*}$, *the origin for (17) system is GAS.*

Consider a network of $N = 24$ Lorenz oscillators with state $x_{k,i} := [\mathrm{x}_{k,i}\ y_{k,i}\ z_{k,i}]^\top$ and dynamics

$$\dot{x}_{k,i} = H_{k,i}(\mathrm{x}_{k,i})x_{k,i}, \quad H_{k,i} := \begin{bmatrix} -\sigma_{k,i} & \sigma_{k,i} & 0 \\ \rho_{k,i} & -1 & -\mathrm{x}_{k,i} \\ 0 & \mathrm{x}_{k,i} & -\beta_{k,i} \end{bmatrix}$$

where $\sigma_{k,i}$, $\beta_{k,i}$, $\rho_{k,i}$ are positive constants. Let these systems be interconnected over a strongly connected directed network that can be compartmentalized into $m = 3$ modules containing each $N_k = 8$ nodes. To enforce global asymptotic stability of the origin, we set

$$\dot{x}_{k,9} = \begin{bmatrix} -\alpha \mathrm{x}_{k,9} & -\alpha y_{k,i} & 0 \end{bmatrix}^\top, \tag{19}$$

$$v_{k,i} = -\sigma^I(x_{k,i} - x_{k,9}) \quad \forall\, i \in \{1, 3\} \tag{20}$$

$v_{k,i} = 0$ for all $i \in \{2,4,5,6,7,8\}$. Then, we obtain the modified averaged dynamics,

$$\dot{x}_e = \begin{bmatrix} -\sigma^* - \alpha\omega_2^* & \sigma^* & 0 \\ \rho^* & -\alpha\omega_2^* - \omega_1^* & -\omega_1^* x_{e1} \\ 0 & \omega_1^* x_{e1} & -\beta^* \end{bmatrix} x_e, \tag{21}$$

where

$$\sigma^* := \sum_{k=1}^{m} \sum_{i=1}^{N_k} w_{\ell k} v_{\ell k i} \sigma_{k,i}, \qquad \rho^* := \sum_{k=1}^{m} \sum_{i=1}^{N_k} w_{\ell k} v_{\ell k i} \rho_{k,i},$$

$$\omega_1^* := \sum_{k=1}^{m} \sum_{i=1}^{N_k} w_{\ell k} v_{\ell k i}, \qquad \omega_2^* := \sum_{k=1}^{m} \sum_{i=N_k+1}^{N_k+N_k'} w_{\ell k} v_{\ell k i}.$$

A straightforward computation, using the Lyapunov function $\mathcal{V}(x_e) = \frac{1}{2}||x_e||^2$, shows that $\dot{\mathcal{V}}(x_e) \leq -q||x_e||^2$, with $q > 0$, for any $\alpha > \max\{\frac{(\rho^*+\sigma^*)^2 - 2\sigma^*}{2\omega_2^*}, \frac{(\rho^*+\sigma^*)^2 - 2\omega_1^*}{2\omega_2^*}\}$.

Illustrative example I

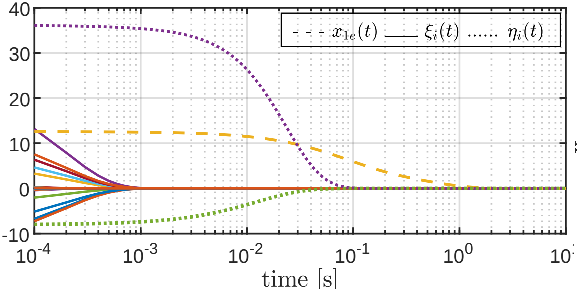

Figure: Trajectories of the closed-loop system in logarithmic time scale. In this simulation we used $\sigma^I = 5000, \sigma^E = 300$

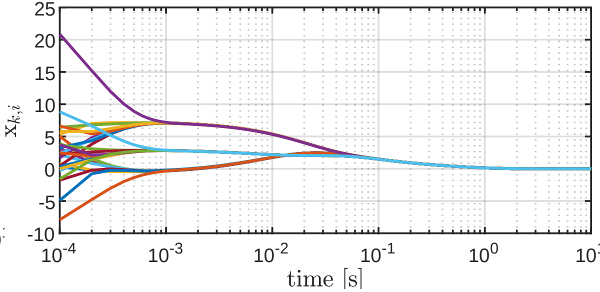

Figure: Trajectories of the closed-loop system in logarithmic time scale, with $\sigma^I = 5000, \sigma^E = 300$.

Illustrative example I

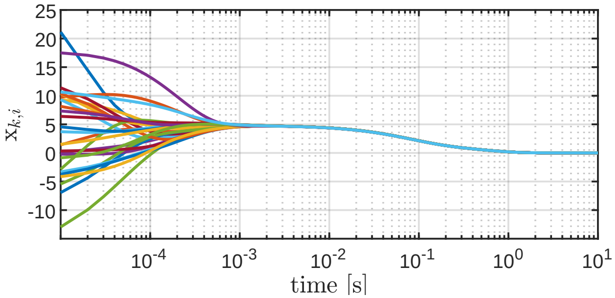

Figure: Trajectories of the closed-loop system in logarithmic time scale, with $\sigma^I = 5000$, $\sigma^E = 12470 > \sigma^I \frac{|L^I|}{|L^E|}$. Since the difference between σ^I and σ^E is relatively small, we observe only a two-time-scale behavior. First, the trajectories synchronize, and then, they converge to zero.

Stabilization via added control modules

Corollary

Consider the networked system $\dot{x} = F(x) - \sigma^I[L^I \otimes I_n]x - \sigma^E[L^E \otimes I_n]x + v(x, x_c)$, *under Assumptions 1 and 2, and the dynamic control extensions*

$$\begin{aligned}\dot{x}_{m+k',i} =& f_{m+k',i}(x_{m+k',i}) - \sigma^I \sum_{l=1}^{N_{m+k'}} b_{m+k',i,l}(x_{m+k',i} - x_{m+k',l}) \\ & - \sigma^E \sum_{k=1}^{m} \sum_{j=1}^{N_k} a''_{(m+k',i),(k,j)}(x_{m+k',i} - x_{k,j})\end{aligned}$$

Assume, in addition, that the origin for $\dot{x}_e = \sum_{k=1}^{m+m'} \sum_{i=1}^{N_k} w_{\ell k} v_{\ell k i} f_{k,i}(x_e)$, *is GAS and there exists a continuously differentiable Lyapunov function satisfying* (8)-(9). *Then, there exist* $\sigma^{E^*} > 0$ *and* $\sigma^{I^*} > 0$ *such that, for all* $\sigma^I > \sigma^{I^*}$ *and* $\sigma^E > \sigma^{E^*}$, *the origin for* (17) *system is GAS.*

Illustrative example II

Let m' denote the number of new modules. For each $k' \leq m'$, the dynamics of the ith node within the k'th module is given by

$$\begin{aligned}\dot{x}_{m+k',i} =& f_{m+k',i}(x_{m+k',i}) - \sigma^I \sum_{l=1}^{N_{m+k'}} b_{m+k',i,l}(x_{m+k',i} - x_{m+k',l}) \\ & - \sigma^E \sum_{k=1}^{m} \sum_{j=1}^{N_k} a''_{(m+k',i),(k,j)}(x_{m+k',i} - x_{k,j}) \end{aligned} \tag{22}$$

where the coefficients $b_{m+k',i,l}$ represent internal interconnections within the $(m+k')$th module, *i.e.*,

$$b_{m+k',i,l} := \begin{cases} 1 & \text{if there is an edge from } (m+k',i) \text{ to } (m+k',l) \\ 0 & \text{if otherwise.} \end{cases}$$

Correspondingly, the coupling$v_{k,i} = -\sigma^E \sum_{k'=1}^{m'} \sum_{j=1}^{N_{m+k'}} a''_{k,k',i,j}(x_{k,i} - x_{m+k',j})$ is added to the existing nodes in the original network.

Illustrative example II

The value of the control parameter α may be computed as for the previous example.

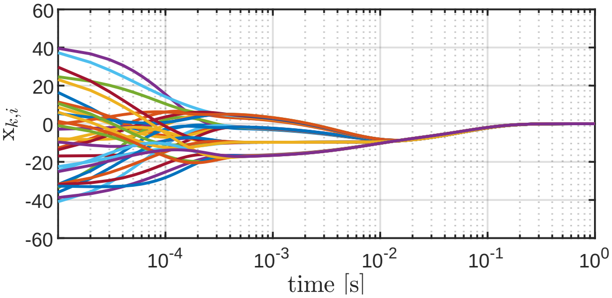

Figure: Trajectories of the closed-loop system in logarithmic time scale, with $\sigma^I = 9000, \sigma^E = 900$. A three-time-scales behavior is appreciated, in view of the large discrepancy between σ^I and σ^E.

Conclusion

Conclusion

- The study can be extended to the analysis of orbital stability for directed modular networks in the context of synchronization of oscillators.
- An important extension yet to establish concerns networks under output coupling and output synchronization.
  Published papers on this topic:

- Anes Lazri, Elena Panteley, Antonio Loria. On Global Asymptotic Stability of Heterogeneous Modular Networks with Three Time-Scales. 9th Int. Conf. on Control, Decision, and Information Technologies, May 2023, Rome, Italy. pp.592-597, ⟨10.1109/CoDIT58514.2023.10284134⟩. ⟨hal-04049681v2⟩

- Anes Lazri, Elena Panteley, Antonio Loria. Analysis and Control of Multi-timescale Modular Directed Heterogeneous Networks. 2023. ⟨hal-04266142⟩

- Adhikari, B., E. Panteley, and I.-C. Morărescu (2022). "Three Time Scales Modeling of the Undirected Clustered Network". In: *2022 IEEE 61st Conference on Decision and Control (CDC)*. DOI: 10.1109/CDC51059.2022.9992973.
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- Panteley, E. and A. Loria (2017). "Synchronization and Dynamic Consensus of Heterogeneous Networked Systems". In: *IEEE Trans. on Automatic Control* 62.8. DOI: 10.1109/TAC.2017.2649382.
- Romeres, Diego, Florian Dörfler, and Francesco Bullo (2013). "Novel results on slow coherency in consensus and power networks". In: *2013 European Control Conference (ECC)*, pp. 742–747. DOI: 10.23919/ECC.2013.6669400.
- Varma, Vineeth, Irinel-Constantin Morarescu, and yezekael Hayel (2018). "Analysis and control of multi-leveled opinions spreading in social networks". In: *American Control Conference, ACC 2018*.