Quantum mean-field filtering and control

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Contents

Main Motivation

- Physical motivation :
  - Mean-field approximation to reduce the complexity.
- Mathematical motivation :
  - Mean-field Games (MFG) and Mean-field Control (MFC) for quantum particles ?

MFG and MFC

- Initiated by Lasry-Lions and independently by Huang-Caines-Malhamé in 2006.
- Theory to analysis differential games with very large number of agents.

Remark : In this talk agent = player = particle.

N-particles systems

- N-controlled particles each one taken value in **space state** $\mathfrak{X} = \mathbb{R}, [0,1], \mathcal{M}, \ldots$
- The **space state** of the system is the Cartesian product $\mathfrak{X}^N = \mathfrak{X} \times \mathfrak{X} \times \cdots \times \mathfrak{X}$
- A system of SDE gives the dynamics of each particles.

$$\mathrm{d}X_t^{u,j} = b(X_t^{u,j}, u_t^j, \hat{\nu}_t^{u,N})\mathrm{d}t + \sigma(X_t^{u,j})\mathrm{d}W_t^j, \quad 1 \leq j \leq N$$

$$\hat{\nu}_t^{u,N} := \frac{1}{N}\sum_{k=1}^{N} \delta_{X_t^{u,k}}, \quad \hat{\nu}^{u,N} = (\hat{\nu}_t^{u,N})_{0 \leq t \leq T}, \; u_t^j := u_t(X_t^{u_j,j})$$

- Provided some set $\mathcal{U}$ of admissible strategies and a time horizon T, agent j aims to minimize its cost $\mathcal{U} \ni u \mapsto \mathcal{J}_j(u) \in \mathbb{R}$ with $\mathcal{J}_j(u) \equiv \mathcal{J}(u, \hat{\nu}_t^{u,N})$:

$$\mathcal{J}_j(u) := \mathbb{E}\left[\int_0^T f(X_t^{u_j,j}, u_t^j, \hat{\nu}_t^{u,N})\mathrm{d}t + g(X_T^{u_j,j}, \hat{\nu}_T^{u,N})\right],$$

  - $u_j := (u^1, \ldots, u^{j-1}, u, u^{j+1}, \ldots, u^N)$ and f, g are some cost functions.

N-Particles Problem

- Namely, $u^* := (u^{*,1}, \dots, u^{*,N}) \in \mathcal{U}^N$ is said to achieve a Nash equilibrium if

$$\mathcal{J}(u^{*,j}, \hat{\nu}^{u^*,N}) = \inf_{u \in \mathcal{U}} \mathcal{J}(u, \hat{\nu}^{u_j^*,N}),$$

where $u_j^* := (u^{*,1}, \dots, u^{*,j-1}, u, u^{*,j+1}, \dots, u^{*,N})$.

- **Problem :** Very difficult to solve !

Propagation of chaos

- **Ansatz :**
  - When $N \to \infty$ the particles becomes independent.

$$\mathsf{Law}(X_t^1, \ldots, X_t^N) \approx \nu_t \otimes \cdots \otimes \nu_t.$$

  - We can focus on the typical particle :

$$\mathrm{d}X_t = b(X_t, u_t, \mathcal{L}(X_t))\mathrm{d}t + \sigma(X_t)\mathrm{d}W_t.$$

MFG Problem

- The MFG problem is defined as follows : find a pair $(\hat{u}, \nu)$ such that denoting by $\hat{X}$ the solution of :

$$\mathrm{d}\hat{X}_t = b(\hat{X}_t, \hat{u}(\hat{X}_t), \nu_t)\mathrm{d}t + \sigma(\hat{X}_t)\mathrm{d}W_t$$
$$\hat{X}_0 = x_0.$$

  - then $\nu_t = \mathcal{L}(\hat{X}_t)$, and $\forall t \in [0, T]$

$$\mathcal{J}\big(\hat{u}(\bullet), \nu(\bullet)\big) \leq \mathcal{J}\big(u(\bullet), \nu(\bullet)\big), \quad \forall u \in \mathcal{U}.$$

- From the MFG problem solution, get an approximation for N-particles problems.

Main ingredients

1. Differential games $\rightarrow$ Continuous time + Control by Closed-Loop Feedback
2. Weak interaction $\rightarrow$ Mean Field Approximation $\rightarrow$ Typical particle.

Quantum Mechanics in 1 Slide

- Two concepts in quantum mechanics :
  1. **Superposition**
  2. **Entanglement**
- Evolution and measurement
  1. Evolution of state : Schrödinger equation
  2. Measurement : Born Rule
- Main difficulties :
  1. We can't separate dynamics of each particle.
  2. It's not clear what can be the empirical measure in quantum setting.
  3. We can't measure without distrub the system and continuously observing freeze the dynamics.

- Should consider open quantum systems subject to indirect measurements.
- The formalism was invented through the pioneering work of Belavkin in the '80s.
- The mean-field extension as well as Quantum MFG was done in 2020 by Kolokoltsov.

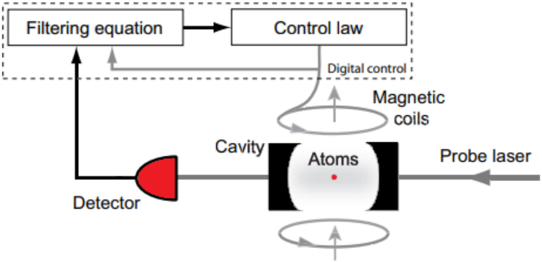

Figure: Synoptic of Quantum Feedback Control

Continuous monitored N-Quantum particle systems

- Focus on finite space state i.e $\mathfrak{X} = \{1, \ldots, d\}$ endowed with Borel measure μ
  - In this case $\mathbb{H} := L^2((\mathfrak{X}, \mu); \mathbb{C}) = \mathbb{C}^d$
  - The state of the system is the Tensor product $\mathbb{H}^N = \mathbb{H} \otimes \mathbb{H} \cdots \otimes \mathbb{H}$
- The state is given by density matrix
  - $\boldsymbol{\rho^N} \in \mathcal{S}(\mathbb{H}^N) := \left\{\rho \in \mathcal{M}_{d^N}(\mathbb{C});\ \rho \geq 0, tr(\rho) = 1, \rho = \rho^\dagger\right\}$
- Evolution of Schrödinger-Belavkin equation :

$$
\begin{aligned}
\mathrm{d}\boldsymbol{\rho}_t^N = -\mathrm{i}[\mathbf{H}, \boldsymbol{\rho}_t^N]\mathrm{d}t + \sum_{j=1}^N \left(\mathbf{L}_j \boldsymbol{\rho}_t^N \mathbf{L}_j^\dagger - \frac{1}{2}\{\mathbf{L}_j^\dagger \mathbf{L}_j, \boldsymbol{\rho}_t^N\}\right)\mathrm{d}t \\
+ \sqrt{\eta} \sum_{j=1}^N \left(\boldsymbol{\rho}_t^N \mathbf{L}_j^\dagger + \mathbf{L}_j \boldsymbol{\rho}_t^N - tr\left((\mathbf{L}_j + \mathbf{L}_j^\dagger)\boldsymbol{\rho}_t^N\right)\boldsymbol{\rho}_t^N\right)\mathrm{d}W_t^j
\end{aligned}
$$

- Signal process :

$$
\mathrm{d}Y_t^j = \mathrm{d}W_t^j + \sqrt{\eta}\, tr\left((\mathbf{L}_j + \mathbf{L}_j^\dagger)\boldsymbol{\rho}_t^N\right)\mathrm{d}t, \quad 1 \leq j \leq N.
$$

Instruments and Operators

- $\mathbf{H} = \sum_j (\mathbf{H}_j + u(Y_t^j)\hat{\mathbf{H}}_j) + \sum_{i<j} \mathbf{A}_{ij}/N.$
- $[C, D] = CD - DC$ and $\{C, D\} = CD + DC$
- $\boldsymbol{C}_j = I \otimes \cdots \otimes C \otimes \cdots \otimes I$
- $\eta \in [0, 1]$ efficiency measurement.
- $H_j = H_j^\dagger$: Free hamiltonien, $\hat{H} = \hat{H}^\dagger$: Controlled Hamiltonian
- u scalar control function.
- Associate measurement operator $L \in \mathcal{M}_{d^N}(\mathbb{C})$ for each particle.
- Pairwise interaction between particles : A of Hilbert-Schmidt form with kernel a i.e,

$$A : L^2(\mathfrak{X}^2; \mathbb{C}) \to L^2(\mathfrak{X}^2; \mathbb{C})$$

$$Af((x, y)) := \int_{\mathfrak{X}^2} a(x, y, x', y') f(x', y') \mu(\mathrm{d}x') \mu(\mathrm{d}y'), \ \forall f \in L^2(\mathfrak{X}^2; \mathbb{C}).$$

$$a(l, l', k', k) = a(l', l, k, k') \ \text{ and } \ a(l, l', k, k') = \overline{a(l, l', k, k')}$$

Quantum propagation of chaos

- **Ansatz :**
  - The density matrix in QM plays same role of classical distribution

$$\boldsymbol{\rho}_t^N \approx \gamma_t \otimes \cdots \otimes \gamma_t$$

Mean-Field Belavkin equation :

$$\begin{aligned} \mathrm{d}\gamma_t &= (-\mathrm{i}[H + u(\gamma_t)\hat{H} + A^{m_t}, \gamma_t])\mathrm{d}t + \left(L\gamma_t L^\dagger - \frac{1}{2}\{L^\dagger L, \gamma_t\}\right)\mathrm{d}t \\ &\quad + \sqrt{\eta}\Big(\gamma_t L^\dagger + L\gamma_t - tr\big((L + L^\dagger)\gamma_t\big)\gamma_t\Big)\mathrm{d}W_t, \\ \mathrm{d}Y_t &= \mathrm{d}W_t + \sqrt{\eta}\, tr\big((L + L^\dagger)\gamma_t\big)\mathrm{d}t \end{aligned} \tag{1}$$

where

- $m_t := \mathbb{E}[\gamma_t]$,
- $\gamma_0 = \rho_0 \in \mathcal{S}(\mathbb{H})$,
- $A^m(l, l') = \sum_{\mathfrak{X}^2} a(l, l'; k, k')\overline{m(k, k')}$,
- $\eta \in [0, 1]$.

Theorem

Theorem (Well posedness and propagation of Chaos)

Let $T > 0,\ U > 0$*, and let* $u : \mathcal{S}(\mathbb{H}) \to [-U, U]$ *be bounded and Lipschitz, i.e.* $|u(\rho) - u(\rho')| \leq \kappa \, \|\rho - \rho'\|$*, with* $\kappa > 0$*. Then* (1) *is well posed and valued in* $\mathcal{S}(\mathbb{H})$*. Furthermore for* $\eta = 1$*, there exists a constant* $c \equiv c(\|A\|, \|\hat{H}\|, \|L\|)$ *such that*

$$\mathbb{E}\big[\alpha_N(t)\big] \leq e^{ct} \left(\alpha_N(0) + \frac{1}{\sqrt{N}} \right),$$

- Where $\| \cdot \|$ corresponds to any matrix norm.
- $\alpha_N(t) := \alpha_{N,1}(t) = 1 - tr(\boldsymbol{\gamma}_t^1 \boldsymbol{\rho}_t^N)$

Sketch of proof

Well-posdness :

- For each $\xi \in C\big([0,T],\mathcal{S}(\mathbb{H})\big)$, consider

$$\begin{aligned} d\gamma_t^\xi &= -i[F_t^\xi,\gamma_t^\xi]dt + \Big(L\gamma_t^\xi L^\dagger - \frac{1}{2}\{L^\dagger L,\gamma_t^\xi\}\Big)dt \\ &+ \sqrt{\eta}\Big(\gamma_t^\xi L^\dagger + L\gamma_t^\xi - tr\big((L+L^\dagger)\gamma_t^\xi\big)\gamma_t^\xi\Big)dW_t, \end{aligned} \tag{2}$$

  where $F_t^\xi := H + u_t\hat{H} + A^{\xi_t}$.
- The control u in open-loop i.e $u(\gamma_t^\xi) = u_t$

Sketch of proof

Lemma (Linear Belavkin equation)

The following linear equation is well-posed in set of matrices and preserve positivity

$$\mathrm{d}\tilde{\gamma}_t^\xi = -\mathrm{i}[F_t^\xi, \tilde{\gamma}_t^\xi]\mathrm{d}t + \Big(L\tilde{\gamma}_t^\xi L^\dagger - \frac{1}{2}\{L^\dagger L, \tilde{\gamma}_t^\xi\}\Big)\mathrm{d}t + \sqrt{\eta}\Big(\tilde{\gamma}_t^\xi L^\dagger + L\tilde{\gamma}_t^\xi\Big)\mathrm{d}Y_t, \tag{3}$$

Moreover equation (2) *has unique solution* $\gamma_t^\xi = \frac{\tilde{\gamma}^\xi}{tr(\tilde{\gamma}^\xi)}$ [a]

[a] Set of nonnegative nonzero matrices is invariant for $\tilde{\gamma}$, which implies that $tr(\tilde{\gamma}) > 0, \forall t$

Lemma

Equation (2) *with* $u_t = u(\gamma_t)$, $u \in C^1$, *and* $\gamma_0^\xi = \gamma_0 \in \mathcal{S}(\mathbb{H})$ *has a unique strong solution.*

Sketch of proof

- From the existence of the family of equations parametrized by ξ, define the mapping $\Xi : C\big([0,T],\mathcal{S}(\mathbb{H})\big) \to C\big([0,T],\mathcal{S}(\mathbb{H})\big)$ by $\Xi(\xi) := (\mathbb{E}[\gamma_t^{\xi}])_{0\leq t\leq T}$.
  - The process γ^m corresponds to the solution of (1) if and only if $m = \Xi(m)$.
    - The existence and uniqueness by showing that the mapping Ξ has a unique fixed point i.e the map Ξ is a contraction with respect to the uniform norm on $C\big([0,T],\mathcal{S}(\mathbb{H})\big)$.

Sketch of proof - Part 2

Propogation of chaos :
The main measures of the deviation of the solutions of $(\boldsymbol{\rho}_t^N)_{t\geq 0}$, to N-particle systems from the product of the solutions to the MF-equation are the following :

$$\mathcal{E}_N^J(t) := 1 - tr(\boldsymbol{\gamma}_t^J \boldsymbol{\rho}_t^J)$$

Where, $J \subset J_N := \{1, \ldots, N\}$ we denote
$\boldsymbol{\rho}^J = tr_{J_N/J}(\boldsymbol{\rho}^N)$ and $\boldsymbol{\gamma}^J = \bigotimes_{j\in J} \gamma$

Lemma (Pickl (2011) - Kolokoltsov (2022))

$$\mathbb{E}\left[\mathcal{E}_N^J(t)\right] \leq |J|\mathbb{E}[\alpha_N(t)]$$

where,

$$\alpha_{N,j}(t) = 1 - tr\left(\gamma_t \rho_t^j\right) = 1 - tr(\boldsymbol{\gamma}_t^j \boldsymbol{\rho}_t^N)$$

here, $\boldsymbol{\gamma}^j := \mathbf{I} \otimes \cdots \otimes \gamma \otimes \cdots \otimes \mathbf{I}.$

Sketch of proof - Part 2

- Very briefly use Itô formula : $\boldsymbol{\gamma} := \boldsymbol{\gamma}^j$, $\mathbf{L} := \mathbf{L}_j$. By Itô's formula, get

$$\begin{aligned} \mathrm{d}\alpha_N(t) &= -tr\big(\mathrm{d}\boldsymbol{\rho}_t^N \boldsymbol{\gamma}_t\big) - tr\big(\boldsymbol{\rho}_t^N \mathrm{d}\boldsymbol{\gamma}_t\big) - tr\big(\mathrm{d}\boldsymbol{\rho}_t^N \mathrm{d}\boldsymbol{\gamma}_t\big). \\ &= \big(P_t^{(1)} + P_t^{(2)}\big)\mathrm{d}t + \sum_k P_t^{(3,k)} \mathrm{d}W_t^k, \end{aligned}$$

$$P_t^{(1)} = \mathrm{i}tr\Big([\frac{1}{N}\sum_{k\neq j}\mathbf{A}_{kj} - \mathbf{A}_j^{m_t} + (u(\rho_t^j) - u(\gamma_t))\hat{\mathbf{H}}, \mathbf{I} - \boldsymbol{\gamma}_t]\boldsymbol{\rho}_t^N\Big)$$

$$\begin{aligned} P_t^{(2)} =& -tr(\boldsymbol{\gamma}_t \mathbf{L}\boldsymbol{\rho}_t^N \mathbf{L}^\dagger + \boldsymbol{\gamma}_t \mathbf{L}^\dagger \boldsymbol{\rho}_t^N \mathbf{L} + \boldsymbol{\gamma}_t \mathbf{L}^\dagger \boldsymbol{\rho}_t^N \mathbf{L}^\dagger + \boldsymbol{\gamma}_t \mathbf{L} + \boldsymbol{\rho}_t^N \mathbf{L}) \\ &+ \Big[tr(\boldsymbol{\gamma}_t \boldsymbol{\rho}_t^N \mathbf{L}^\dagger + \boldsymbol{\gamma}_t \mathbf{L}\boldsymbol{\rho}_t^N) tr(\boldsymbol{\gamma}_t(\mathbf{L}^\dagger + \mathbf{L})) + \\ & tr(\boldsymbol{\gamma}_t \boldsymbol{\rho}_t^N \mathbf{L}^\dagger + \boldsymbol{\gamma}_t L\boldsymbol{\rho}_t^N) tr(\boldsymbol{\rho}_t^N(\mathbf{L}^\dagger + \mathbf{L})) - \\ & tr(\boldsymbol{\rho}_t^N \boldsymbol{\gamma}_t) tr(\boldsymbol{\rho}_t^N(\mathbf{L}^\dagger + \mathbf{L})\Big) tr(\boldsymbol{\gamma}_t(\mathbf{L} + \mathbf{L}^\dagger))\Big], \end{aligned}$$

  and $P_t^{(3,k)}$ are bounded quantities.
- By taking an expectation of the above equation and using several fastidious estimates lemmas,

$$\frac{\mathrm{d}\mathbb{E}[\alpha_N(t)]}{\mathrm{d}t} \leq \big(C\|A\| + \kappa\|\hat{H}\|\big)\mathbb{E}[\alpha_N(t)] + \frac{C}{\sqrt{N}} + C'\|L\|^2\mathbb{E}[\alpha_N(t)].$$

  The proof is fulfiled by classical Gronwall inequality.

Application - Stabilization

- Consider the case of N-qubit system [1]. With $A = a_1^\dagger a_2 + a_2^\dagger a_1$. [2] $A(l, l'; k, k')$ such that $A(2,1;1,2) = A(1,2;2,1) = 1$ and zeros otherwise. For each particle we associate a free Hamiltonian $\mathbf{H}_j = \boldsymbol{\sigma_z}^j$, an observation channel $\mathbf{L}_j = \boldsymbol{\sigma_z}^j$ and a controlled Hamiltonian $\hat{\mathbf{H}}_j = \boldsymbol{\sigma_x}^j$.

$$
\begin{aligned}
\mathrm{d}\boldsymbol{\rho}_t^N = & -\mathrm{i}[\mathbf{H}_t, \boldsymbol{\rho}_t^N]\mathrm{d}t + \sum_{j=1}^N \left(\boldsymbol{\sigma_z}^j \boldsymbol{\rho}_t^N \boldsymbol{\sigma_z}^j - \boldsymbol{\rho}_t^N \right)\mathrm{d}t \\
& + \sqrt{\eta} \sum_{j=1}^N \left(\boldsymbol{\rho}_t^N \boldsymbol{\sigma_z}^j + \boldsymbol{\sigma_z}^j \boldsymbol{\rho}_t^N - 2tr\left(\boldsymbol{\sigma_z}^j \boldsymbol{\rho}_t^N\right)\boldsymbol{\rho}_t^N \right)\mathrm{d}W_t^j .
\end{aligned}
$$

$$
\sigma_x := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \ \sigma_y := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \ \sigma_z := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}
$$

[1] $\mathcal{X} = \{1, 2\}$

[2] Where $a_j^\dagger$ and a_j are the creation and annihilation operators respectively for the j-th qubit.

Application - Stabilization

- For the MF equation straightforward calculations in Pauli basis give us $A^m = \begin{pmatrix} 0 & \mathbb{E}[x] - \mathrm{i}\mathbb{E}[y] \\ \mathbb{E}[x] + \mathrm{i}\mathbb{E}[y] & 0 \end{pmatrix}$.
  MF Belavkin equation projected in Pauli basis is represented as follows:

$$
\begin{aligned}
\mathrm{d}x_t &= \Big(- y_t - x_t + z_t \mathbb{E}[y_t] \Big)\mathrm{d}t - \sqrt{\eta} x_t z_t \mathrm{d}W_t, \\
\mathrm{d}y_t &= \Big(x_t - y_t + u(\gamma_t) z_t - z_t \mathbb{E}[x_t] \Big)\mathrm{d}t + \sqrt{\eta} y_t z_t \mathrm{d}W_t, \\
\mathrm{d}z_t &= \Big(- u(\gamma_t) x_t + y_t \mathbb{E}[x_t] + x_t \mathbb{E}[y_t] \Big)\mathrm{d}t + \sqrt{\eta}(1 - z_t^2)\mathrm{d}W_t.
\end{aligned}
$$

- We note $\{\rho_e, \rho_g\}$

$$
\rho_g := \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \ \rho_e := \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},
$$

  the matrices are the equilibrium points of the MF-equation.

Application - Stabilization

- Studying the asymptotic behavior of system.

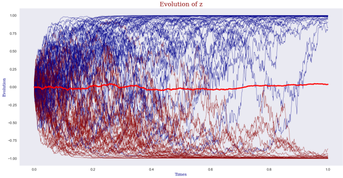

Figure: quantum states reduction, when $u \equiv 0$

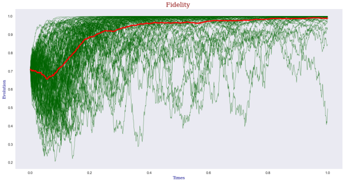

Figure: fidelity with feedback, when $u(\gamma) := -7.6itr\big([\sigma_x, \gamma]\rho_e\big) + 5\big(1 - tr(\gamma\rho_e)\big)$

Further research

1. Establish a rigorous mean-field convergence result in terms of density matrices.
2. Stable numerical scheme based on Krauss Map + Particle Methods
3. Extension of QMFG in heterogenous case $\mapsto$ Graphon Quantum Games ?

The End