LMI-based Suboptimal Consensus for a class of LTI Multi-agent Systems

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Control Design for Multi-agent Systems (MAS)*

- Multi-agent systems control design: global (system level) objectives to be accomplished using local (agent level) interaction.

Information Consensus in Multivehicle Cooperative Control- Wie Ren, Randal W. Beard and Ella M .Atkins IEEE Control Systems Magazine, April-2007.

Control Design for Multi-agent Systems (MAS)*

- Multi-agent systems control design: global (system level) objectives to be accomplished using local (agent level) interaction.
- Natural phenomenon:

Figure: Flocking of birds

Figure: Schooling of fish

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Control Design for MAS

- Idea: Collective group behavior through local interaction.

Control Design for MAS

- Idea: Collective group behavior through local interaction.
- *Due to local interactions and thus the constraints on the communication capabilities, the control input is inherently structured and hence not trivial to compute.*

Control Design for MAS ≡ Structured Input Design

(Undirected) Line Topology

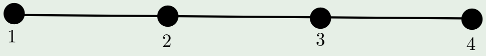

Figure: (Undirected) Line-topology of agent interaction

Control Design for MAS ≡ Structured Input Design

(Undirected) Line Topology

Figure: (Undirected) Line-topology of agent interaction

For this model of interaction with *relative information exchange*, the (linear) inputs for the agents must be of the form

$$\text{Agent } 1 : \mathbf{u}_1(t) = \alpha_{12}(\mathbf{x}_2(t) - \mathbf{x}_1(t)),$$
$$\text{Agent } 2 : \mathbf{u}_2(t) = \alpha_{21}(\mathbf{x}_1(t) - \mathbf{x}_2(t)) + \alpha_{23}(\mathbf{x}_3(t) - \mathbf{x}_2(t)),$$
$$\text{Agent } 3 : \mathbf{u}_3(t) = \alpha_{32}(\mathbf{x}_2(t) - \mathbf{x}_3(t)) + \alpha_{34}(\mathbf{x}_4(t) - \mathbf{x}_3(t)), \text{and}$$
$$\text{Agent } 4 : \mathbf{u}_4(t) = \alpha_{43}(\mathbf{x}_3(t) - \mathbf{x}_4(t)).$$

where α_{ij}s are constants to be computed.

Control Design for LTI MAS

Problem 1 (a class of LTI MAS)

(system) Consider a group of agents with dynamics

$$\dot{\mathbf{x}}_i(t) = A\mathbf{x}_i(t) + B_i\mathbf{u}_i(t);\ \mathbf{x}_i(t_0) = \mathbf{x}_{i0},\ i \in \{1, 2, \dots N\},$$

where $\mathbf{x}_i(t) \in \mathbb{R}^n$*,* $\mathbf{u}_i(t) \in \mathbb{R}^m$*,* B_i *being full rank* $\forall i \in \{1, 2, 3 \dots N\}$*, and* $m \leq n$ *communicating over a given undirected topology.*

Control Design for LTI MAS

Problem 1 (Protocol design for a class of LTI MAS)

(system) Consider a group of agents with dynamics

$$\dot{\mathbf{x}}_i(t) = A\mathbf{x}_i(t) + B_i\mathbf{u}_i(t);\; \mathbf{x}_i(t_0) = \mathbf{x}_{i0},\; i \in \{1, 2, \ldots N\},$$

where $\mathbf{x}_i(t) \in \mathbb{R}^n$, $\mathbf{u}_i(t) \in \mathbb{R}^m$, B_i *full rank* $\forall i \in \{1, 2, 3 \ldots N\}$, *and* $m \le n$ *communicating over a given undirected topology.*

(protocol) Design a control law of the form

$$\mathbf{u}_i(t) = \sum_{j \in \mathcal{N}_i} K_{ij} \left(\mathbf{x}_i(t) - \mathbf{x}_j(t) \right), i \in \{1, 2, 3, \ldots\ldots, N\}$$

$\mathcal{N}_i$ *denoting the neighborhood of agent* i *that is set of all agents interacting with agent* i,

Consensus Protocol Design for LTI MAS

Problem 1 (Consensus Protocol design for a class of LTI MAS)

(system) Consider a group of agents with dynamics

$$\dot{\mathbf{x}}_i = A\mathbf{x}_i + B_i\mathbf{u}_i;\ \mathbf{x}_i(t_0) = \mathbf{x}_{i0},\ i \in \{1, 2, \ldots N\},$$

where $\mathbf{x}_i \in \mathbb{R}^n$, $\mathbf{u}_i \in \mathbb{R}^m$, B_i *is full rank* $\forall i \in \{1, 2, 3 \ldots N\}$, *and* $m \leq n$ *communicating over a given undirected topology.*
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$$\mathbf{u}_i(t) = \sum_{j \in \mathcal{N}_i} K_{ij} \left(\mathbf{x}_i(t) - \mathbf{x}_j(t)\right), i \in \{1, 2, 3, \ldots\ldots, N\}$$

$\mathcal{N}_i$ *denoting the neighborhood of agent* i *that is set of all agents interacting with agent* i, *such that*

(objective) **consensus (state agreement)** *is achieved that is* $\|\mathbf{x}_i(t) - \mathbf{x}_j(t)\| \to 0$ *as* $t \to \infty$ $\forall i \neq j$ *and* $i, j \in \{1, 2, 3, \ldots\ldots, N\}$.

Suboptimal Consensus Protocol Design for LTI MAS

Problem 1 (Suboptimal Consensus Protocol design for a class of LTI MAS)

(system) Consider a group of agents with dynamics

$$\dot{\mathbf{x}}_i(t) = A\mathbf{x}_i(t) + B_i\mathbf{u}_i(t);\ \mathbf{x}_i(t_0) = \mathbf{x}_{i0},\ i \in \{1, 2, \dots N\},$$

where $\mathbf{x}_i(t) \in \mathbb{R}^n$, $\mathbf{u}_i(t) \in \mathbb{R}^m$, B_i *full rank* $\forall i \in \{1, 2, 3 \dots N\}$, *and* $m \leq n$ *communicating over a given undirected topology.*
(protocol) Design a control law of the form

$$\mathbf{u}_i(t) = \sum_{j \in \mathcal{N}_i} K_{ij} \left(\mathbf{x}_i(t) - \mathbf{x}_j(t)\right), i \in 1, 2, 3, \dots\dots, N$$

$\mathcal{N}_i$ *denoting the neighborhood of agent* i *that is set of all agents interacting with agent* i, *such that*
(objective) **consensus (state agreement)** *is achieved that is* $\mathbf{x}_i(t) - \mathbf{x}_j(t) \to \mathbf{0}$ *as* $t \to \infty$ $\forall i \neq j$ *and* $i, j \in \{1, 2, 3, \dots\dots, N\}$.
(quantify suboptimality) Also, determine the upper bound $\gamma > 0$ *on the cost*

$$\hat{J} = \int_{t_0}^{\infty} \frac{1}{2} \left(\sum_{i=1}^{N} \sum_{j \in \mathcal{N}_i} \left(\mathbf{x}_i - \mathbf{x}_j\right)^T Q_{ij} \left(\mathbf{x}_i - \mathbf{x}_j\right) + \sum_{i=1}^{N} \mathbf{u}_i^T R_i \mathbf{u}_i \right) dt$$

where $Q_{ij} = Q_{ij}^T = \underline{Q} \geq \mathbf{0}\ \forall i, j$, $R_i = R_i^T = \underline{R} \succ \mathbf{0}\ \forall i$.

Translation to Structured Suboptimal LQR

Error Dynamics

Define a new state vector for the i–th agent as

$$\mathbf{e}_i = \mathbf{x}_i - \mathbf{x}_{i+1}, i = 1, \ldots, N-1. \tag{1a}$$

The overall dynamics of the multiagent system can be equivalently represented by

$$\dot{\mathbf{e}} = \tilde{A}\mathbf{e} + \tilde{B}\hat{\mathbf{u}}, \mathbf{e}(t_0) = \mathbf{e}_0, \tag{1b}$$

where $\mathbf{e} = \mathtt{col}(\mathbf{x}_1 - \mathbf{x}_2, \mathbf{x}_2 - \mathbf{x}_3, \ldots, \mathbf{x}_{N-1} - \mathbf{x}_N) \in \mathbb{R}^{n(N-1)}, \hat{\mathbf{u}} = \mathtt{col}(\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_N) \in \mathbb{R}^{mN}, \tilde{A} = I_{N-1} \otimes A$ and

$$\tilde{B} = \begin{bmatrix} B_1 & -B_2 & \cdots & \cdots & \cdots & \mathbf{0} \\ \mathbf{0} & B_2 & -B_3 & \cdots & \cdots & \mathbf{0} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \cdots & B_{N-1} & -B_N \end{bmatrix}. \tag{1c}$$

Translation to Structured Suboptimal LQR

Suboptimal Structured LQR

Design an optimal/suboptimal control law $\hat{\mathbf{u}} = -K^e\mathbf{e}$ such that

1. K^e has the desired structure (as imposed by the communication topology), and
2. Consensus is achieved i.e. $\mathbf{e}_i \to \mathbf{0}\ \forall i \in \{1,2,3..,N-1\}$(regulation of errors) and the (quadratic) cost $\hat{J} = \int_0^\infty \left(\mathbf{e}^T\tilde{Q}\mathbf{e} + \hat{\mathbf{u}}\tilde{R}\hat{\mathbf{u}}\right)dt$ is minimized.

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(Note that $\tilde{Q}$ and $\tilde{R}$ are readily obtained from $\underline{Q}$ and $\underline{R}$.)

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K^e is chosen to be of the form $K^e = -\tilde{R}^{-1}\tilde{B}^T\hat{P}$. †

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(Note that $\tilde{Q}$ and $\tilde{R}$ are readily obtained from $\underline{Q}$ and $\underline{R}$.)

K^e is chosen to be of the form $K^e = -\tilde{R}^{-1}\tilde{B}^T\hat{P}$. †

Denote $\mathbb{P}$ as the set constituting the $\hat{P}$ matrices having the desired structure (as imposed by K^e).

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Suboptimal LQR Design

$\epsilon-$ Suboptimal LQR

Consider the system $\dot{\mathbf{x}}(t) = A\mathbf{x}(t) + B\mathbf{u}(t)$ and the associated quadratic cost

$$J(\mathbf{x}(t), \mathbf{u}(t)) = \frac{1}{2}\int_{t_0}^{\infty} \left(\mathbf{x}^T(t)Q\mathbf{x}(t) + \mathbf{u}^T(t)R\mathbf{u}(t)\right)dt.$$

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If there exists a matrix P and a symmetric matrix $\bar{P} \succ \mathbf{0}$ satisfying

$$\begin{bmatrix} A^T P + PA + Q & \left(P + P^T\right)B \\ B^T\left(P + P^T\right) & 4R \end{bmatrix} \succ \mathbf{0},$$

$$\begin{bmatrix} -A^T\bar{P} - \bar{P}A - Q + \eta I & \left(\frac{1}{2}\left(P + P^T\right) - \bar{P}\right)B \\ \left(\frac{1}{2}\left(P + P^T\right) - \bar{P}\right)^T B & R \end{bmatrix} \succ \mathbf{0},$$

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If there exists a matrix P and a symmetric matrix $\bar{P} > \mathbf{0}$ satisfying

$$\begin{bmatrix} A^T P + PA + Q & \left(P + P^T\right) B \\ B^T \left(P + P^T\right) & 4R \end{bmatrix} > \mathbf{0},$$

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such that $A - \frac{1}{2} B R^{-1} \left(P + P^T\right)$ is Hurwitz, then the control law $\mathbf{u} = -\frac{1}{2} R^{-1} B^T \left(P + P^T\right) \mathbf{x}$ is ϵ-suboptimal,

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$$\left(A - \frac{1}{2}BR^{-1}B^T\left(P + P^T\right)\right)^T\tilde{P} + \tilde{P}\left(A - \frac{1}{2}BR^{-1}B^T\left(P + P^T\right)\right) + I = \mathbf{0}.$$

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$$J(\mathbf{x}(t), \mathbf{u}(t)) = \frac{1}{2}\int_{t_0}^{\infty}\left(\mathbf{x}^T(t)Q\mathbf{x}(t) + \mathbf{u}^T(t)R\mathbf{u}(t)\right)dt.$$

If there exists a matrix P and a symmetric matrix $\bar{P} > \mathbf{0}$ satisfying

$$\begin{bmatrix} A^TP + PA + Q & \left(P + P^T\right)B \\ B^T\left(P + P^T\right) & 4R \end{bmatrix} > \mathbf{0},$$

$$\begin{bmatrix} -A^T\bar{P} - \bar{P}A - Q + \eta I & \left(\frac{1}{2}\left(P + P^T\right) - \bar{P}\right)B \\ \left(\frac{1}{2}\left(P + P^T\right) - \bar{P}\right)^T B & R \end{bmatrix} > \mathbf{0},$$

such that $A - \frac{1}{2}BR^{-1}\left(P + P^T\right)$ is Hurwitz, then the control law $\mathbf{u} = -\frac{1}{2}R^{-1}B^T\left(P + P^T\right)\mathbf{x}$ is ϵ-suboptimal, that is the associated cost satisfies $J < J^* + \epsilon$, with $\epsilon = \mathbf{x}_0^T\left(\bar{P} + \eta\tilde{P}\right)\mathbf{x}_0 - J^*$, J^* being the optimal cost, and $\tilde{P}$ being the solution of the following equation.

$$\left(A - \frac{1}{2}BR^{-1}B^T\left(P + P^T\right)\right)^T\tilde{P} + \tilde{P}\left(A - \frac{1}{2}BR^{-1}B^T\left(P + P^T\right)\right) + I = \mathbf{0}.$$

😇 *A remark on the origin of these LMIs at the end!!*

Suboptimal Consensus Protocol Design

γ- Suboptimal Consensus Protocol Design

Consider the convex optimization problem

$$\min_{\hat{P},\hat{\bar{P}}\succeq\mathbf{0}} \ \eta$$

$$\text{s.t. } \hat{\bar{P}} \in \mathbb{P},$$

$$\left[\begin{array}{cc|cc} \bar{\Gamma}(\hat{P}) & \hat{P}\tilde{B} & \multicolumn{2}{c}{\mathbf{0}} \\ \left(\hat{P}\tilde{B}\right)^T & \hat{R} & & \\ \hline \multicolumn{2}{c|}{\mathbf{0}} & -\bar{\Gamma}(\hat{\bar{P}}) + \eta I & \left(\hat{P} - \hat{\bar{P}}\right)\tilde{B} \\ & & \left(\left(\hat{P} - \hat{\bar{P}}\right)\tilde{B}\right)^T & \hat{R} \end{array}\right] \succ \mathbf{0},$$

where $\bar{\Gamma}(\hat{P}) \triangleq \tilde{A}^T\hat{P} + \hat{P}\tilde{A} + \tilde{Q}$. Let the triplet $\{\hat{P}, \hat{\bar{P}}, \eta\}$ be a solution this problem such that $\left(\tilde{A} - \tilde{B}\hat{R}^{-1}\tilde{B}^T\hat{P}\right)$ is Hurwitz, then the control input $\hat{\mathbf{u}} = K^{\mathbf{e}}\mathbf{e} = -\hat{R}^{-1}\tilde{B}^T\hat{P}\mathbf{e}$ is γ- suboptimal that is the cost $J < \gamma$ with $\gamma = \mathbf{e}_0^T\left(\hat{\bar{P}} + \eta\tilde{P}_e\right)\mathbf{e}_0$, where $\tilde{P}_e$ is the unique positive semi-definite solution of $\left(\tilde{A} - \tilde{B}\hat{R}^{-1}\tilde{B}^T\hat{P}\right)^T\tilde{P}_e + \tilde{P}_e\left(\tilde{A} - \tilde{B}\hat{R}^{-1}\tilde{B}^T\hat{P}\right) + I = \mathbf{0}$.

Suboptimal Consensus Protocol Design- Numerical Results

Four Agents- Line Topology

Consider a four-agent system where the single-integrator agents are communicating over the line topology with dynamics $\dot{x}_i = u_i \;, i \in \{1,2,3,4\}$. Design a control input of the form

$$\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} = \begin{bmatrix} \alpha_{12}(x_2 - x_1) \\ \alpha_{21}(x_1 - x_2) + \alpha_{23}(x_3 - x_2) \\ \alpha_{32}(x_3 - x_2) + \alpha_{34}(x_4 - x_3) \\ \alpha_{43}(x_3 - x_4) \end{bmatrix}$$

such that consensus is achieved and determine γ such that the cost

$$J = \int_0^\infty \left((x_1 - x_2)^2 + (x_2 - x_3)^2 + (x_3 - x_4)^2 + u_1^2 + u_2^2 + u_3^2 + u_4^2 \right) \, dt$$

satisfies $J < \gamma$. Here, α_{ij}s are the design variables. Also, $x_1(0) = 0.1$, $x_2(0) = 0.2$, $x_3(0) = 0.5$ and $x_4(0) = -0.5$.

Suboptimal Consensus Protocol Design- Numerical Results

The solution of the formulated convex optimization problem is found to be $\eta = 1$ with

$$\hat{P} = diag(0.39, 0.37, 0.39).$$

The corresponding feedback-gain matrix is computed to be

$$K^{\mathbf{e}} = \begin{bmatrix} -0.39 & 0 & 0 \\ 0.39 & -0.37 & 0 \\ 0 & 0.37 & -0.39 \\ 0 & 0 & 0.39 \end{bmatrix}.$$

yielding $\alpha_{12} = \alpha_{21} = 0.39$, $\alpha_{23} = \alpha_{32} = 0.37$, and $\alpha_{34} = \alpha_{43} = 0.39$.
Also, $\mathbf{e}_0^T \left(\hat{\bar{P}} + \eta \tilde{P}_e \right) \mathbf{e}_0 = 1.1 = \gamma > J$.
The cost is computed to be $J = 0.89$.

Suboptimal Consensus Protocol Design- Numerical Results

In the existing results , the upper bound γ is *a priori* specified based on which a set of initial conditions of agents is determined. Furthermore, the feedback-gain matrix of the following form is computed

$$K^{\mathbf{e}} = \begin{bmatrix} -c & 0 & 0 \\ c & -c & 0 \\ 0 & c & -c \\ 0 & 0 & c \end{bmatrix},$$

where $c = 1.31$[‡] is computed by solving a Lyapunov equation of the order one (order of the agent). Thus, a uniform gain is computed for all the agents, contrariwise the proposed approach.

‡ Jiao *et. al.*, A suboptimality approach to distributed linear quadratic optimal control, IEEE TAC'.19.

Suboptimal Consensus Protocol Design- Numerical Results

Comparison with existing results in literature

Table: Comparison with existing results

Controller	Feedback-gain matrix	Cost
Centralized (Optimal)	$K^{\mathbf{e}} = \begin{bmatrix} -0.82 & -0.27 & -0.11 \\ 0.54 & -0.65 & -0.16 \\ 0.16 & 0.65 & -0.54 \\ 0.11 & 0.27 & 0.82 \end{bmatrix}$	$J = 0.74$
Proposed	$K^{\mathbf{e}} = \begin{bmatrix} -0.39 & 0 & 0 \\ 0.39 & -0.37 & 0 \\ 0 & 0.37 & -0.39 \\ 0 & 0 & 0.39 \end{bmatrix}$	$J = \mathbf{0.89}$
Existing (as per [§])	$K^{\mathbf{e}} = \begin{bmatrix} -1.31 & 0 & 0 \\ 1.31 & -1.31 & 0 \\ 0 & 1.31 & -1.31 \\ 0 & 0 & 1.31 \end{bmatrix}$	$J = 0.92$

[§] Jiao *et. al.*, A suboptimality approach to distributed linear quadratic optimal control, IEEE TAC'.19.

Suboptimal Consensus Protocol Design- Numerical Results

Error-trajectories

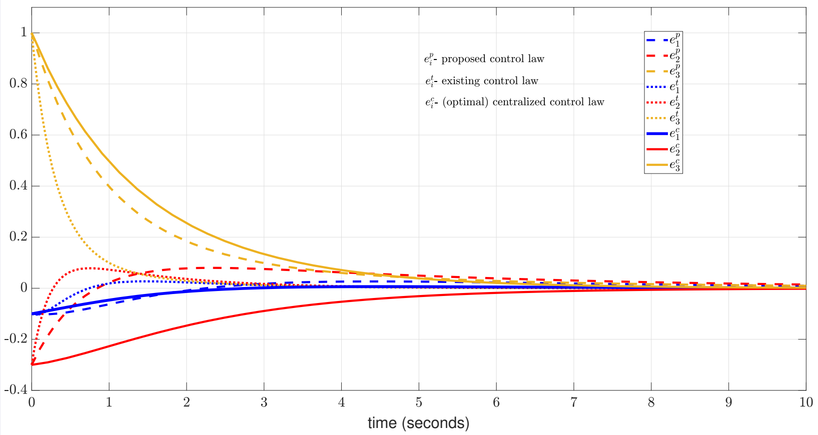

Figure: Error trajectories

Bonus Information commences!

Krotov Sufficient Conditions ¶

Generic Optimal Control Problem (GOCP)

Compute an optimal control law $\mathbf{u}^*(t)$ which minimizes the cost functional:

$$J(\mathbf{x}(t), \mathbf{u}(t)) = l_f(\mathbf{x}(t_f)) + \int_{t_0}^{t_f} l(\mathbf{x}(t), \mathbf{u}(t), t)dt \tag{2}$$

subject to the system dynamics $\dot{\mathbf{x}}(t) = f(\mathbf{x}(t), \mathbf{u}(t), t)$ with $\mathbf{x}(t_0) \in \mathbb{R}^n;\ t \in [t_0, t_f]$.

The state and input vector may be constrained: $\mathbf{x}(t) \in \mathbb{X}$ and $\mathbf{u}(t) \in \mathbb{U}$.

Krotov Conditions

For the GOCP, let $q(\mathbf{x}(t), t)$ (Krotov function) be a *piecewise continuously differentiable function*. Then, there is an equivalent representation of (2) given as below.

$$J_{eq}(\mathbf{x}(t), \mathbf{u}(t)) = s_f(\mathbf{x}(t_f)) + q(\mathbf{x}_0, t_0) + \int_{t_0}^{t_f} s(\mathbf{x}(t), \mathbf{u}(t), t)dt$$

where

$$s(\mathbf{x}(t), \mathbf{u}(t), t) \triangleq \frac{\partial q}{\partial t} + l(\mathbf{x}(t), \mathbf{u}(t), t) + \frac{\partial q}{\partial \mathbf{x}} f(\mathbf{x}(t), \mathbf{u}(t), t),$$

$$s_f(\mathbf{x}(t_f)) \triangleq l_f(\mathbf{x}(t_f)) - q(\mathbf{x}(t_f), t_f).$$

If $\left[\mathbf{x}^*(t), \mathbf{u}^*(t)\right]$ is an *admissible process* such that

$$s(\mathbf{x}^*(t), \mathbf{u}^*(t), t) = \min_{\mathbf{x}\in\mathbb{X}, \mathbf{u}\in\mathbb{U}} s(\mathbf{x}(t), \mathbf{u}(t), t), \forall t \in [t_0, t_f),$$

and

$$s_f(\mathbf{x}^*(t_f)) = \min_{\mathbf{x}\in\mathbb{X}_f} s_f(\mathbf{x});$$

then $\left[\mathbf{x}^*(t), \mathbf{u}^*(t)\right]$ is the optimal process.
Here, $\mathbb{X}_f$ is the admissible terminal set.

Vadim Krotov. Global Methods in Optimal Control Theory. Marcel Dekker, 1995.

Where do LMIs come from?

While utilizing the Krotov framework to infinite-horizon linear quadratic regulation problem, with quadratic Krotov functions $\mathbf{x}^T P \mathbf{x}$ and $\mathbf{x}^T \left(P - \bar{P}\right) \mathbf{x}$, the convexity conditions of the equivalent optimization problem lead to the LMIs used in this work.