# Homogeneous observer for a low-dimensional neural fields model of cortical activity

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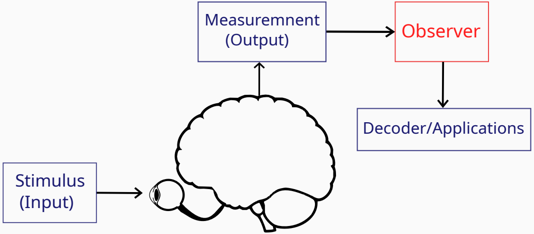
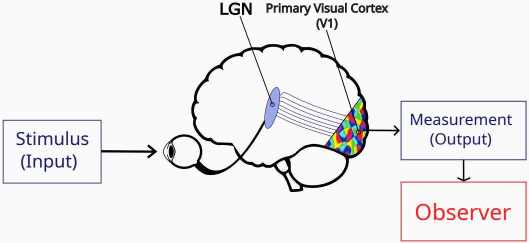
# Motivation and Methodology

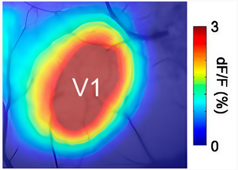



- **Context** : Real-time knowledge of neural activity is key for applications
- **Problem** : Only **partial measurement** is available
- **Objective** : **Estimation** of neural activity from partial measurement
- **Approach** : Formalisation as a control system; **observer** to recover the state from the output

# Formalization

We focus on vision, mainly on the primary visual cortex (V1)




VSD Imaging $\approx V(\cdot, t)$

- We consider as **measurement** the average activity over V1

$$h(V(x,t)) = \frac{1}{|\Omega|}\int_{\Omega} V(x,t)\, dx \qquad \Omega = \text{Area of V1}$$

- Cortical activity is described through a **neural field:**

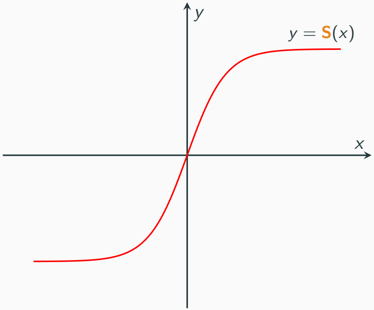
$$\partial_t V(x,t) = -V(x,t) + \int_{\Omega} W(x,x')S(V(x',t))dx' + I_{ext}(x,t)$$

# Formalization

- Cortical activity is described through a **neural field**

$$\partial_t V(x,t) = -V(x,t) + \int_{\Omega} W(x,x')\mathbf{S}\big(V(x',t)\big)dx' + I_{ext}(x,t)$$

- **Non-linear** integro-differential equation

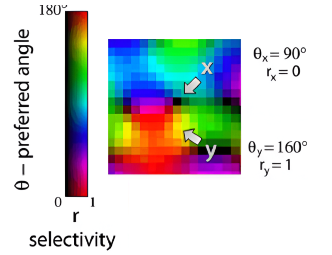



Sigmoid Function

# Model Reduction (Blumenfeld & al)

$$\partial_t V(x,t) = -V(x,t) + \int_{\Omega} W(x,x')S\big(V(x',t)\big)dx' + I_{ext}(x,t)$$

- Neurons of V1 : **selectivity** and **orientation** preference $(r, \theta)$
- Ring model-like connectivity $W(x,y) := w_0 + w_1 r_x r_y \cos(2(\theta_x - \theta_y))$
- Change of variable induced by the polar mapping $\phi(x) = (r_x, \theta_x)$




**Model Reduction**

Under adequate hypothesis on the input $I_{ext}$. All Solutions converges exponentially towards solutions of the following form:

$$V(r,\theta,t) = \; v_0(t) \; + \; v_1(t) r\cos(2\theta) \; + \; v_2(t) r\sin(2\theta)$$

# Model Reduction

The study reduces to the observability and observer design of a 3 dimensional non-linear system for $v = (v_0, v_{1:2}) \in \mathbb{R} \times \mathbb{R}^2$ :

$$\begin{cases} \dot{v}_0 = -v_0 + w_0 \Gamma_0^0(v_0, |v_{1:2}|) + I_0 \\ \dot{v}_{1:2} = -v_{1:2} + w_1 \Gamma_0^1(v_0, |v_{1:2}|) \frac{v_{1:2}}{|v_{1:2}|} + I_{1:2} \qquad v_{1:2} = (v_1, v_2) \\ y = v_0 \end{cases} \qquad (\star)$$

$$\Gamma_0^0(v_0, \rho) = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{+\infty} S\big(v_0 + \rho r \cos 2\theta\big) \frac{P(r)}{\pi} dr d\theta$$

$$\Gamma_0^1(v_0, \rho) = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{+\infty} S\big(v_0 + \rho r \cos 2\theta\big) r \cos(2\theta) \frac{P(r)}{\pi} dr d\theta$$

**Objective**

**Observability analysis** and **Observer Design** for $(\star)$

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**Non observability in the linear case**

If $S(x) = a\, x + b$ **the system is not observable !**

# Observability analysis

$$\begin{cases} \dot{v}_0 = -v_0 + w_0 \Gamma_0^0(v_0, |v_{1:2}|) + I_0 \\ \dot{v}_{1:2} = -v_{1:2} + w_1 \Gamma_0^1(v_0, |v_{1:2}|) \frac{v_{1:2}}{|v_{1:2}|} + I_{1:2} \\ y = v_0 \end{cases} \tag{$\star$}$$

## Observability

Assume $I = (I_0, I_{1:2}) \in C^3([0, +\infty), \mathbb{R} \times \mathbb{R}^2)$

- Assume $I_0 \geq c > 0$. Then $(\star)$ is observable on $[t_1, t_2] \subset \mathbb{R}_+$ if and only if

$$\det\left(I_{1:2}, \dot{I}_{1:2}\right) \not\equiv 0 \qquad \text{on } [t_1, t_2]$$

- If $I_0 = 0$ in any interval $[t_1, t_2] \subset \mathbb{R}_+$, then $(\star)$ is not observable on $[t_1, t_2]$

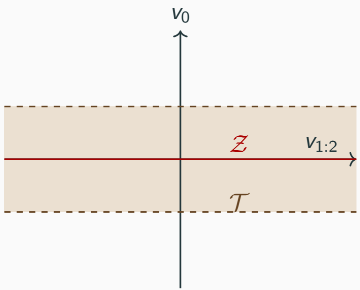
**Singular region for observability** $\mathcal{Z} := \{v_0 = 0\}$. When $v \in \mathcal{Z}$, the system **is not instantaneously observable**

# Observer design approach

**Singularity Crossing**

Suppose $|I_0| > c > 0$. There exist a **transition region** $\mathcal{T} \subset \mathbb{R}^3$ which **contains the singular region** $\mathcal{Z} = \{v_0 = 0\}$ and a time $T > 0$ such that if there exist a time $t_c \geq 0$

$$v(t_c) \in \mathcal{T} \implies v(t) \notin \mathcal{T} \qquad \forall t \geq t_c + T$$



# Observer design approach

**Singularity Crossing**

Suppose $|l_0| > c > 0$. There exist a **transition region** $\mathcal{T} \subset \mathbb{R}^3$ which **contains the singular region** $\mathcal{Z} = \{v_0 = 0\}$ and a time $T > 0$ such that if there exist a time $t_c \geq 0$

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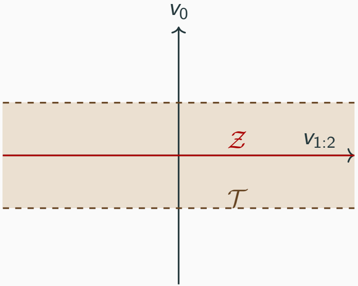
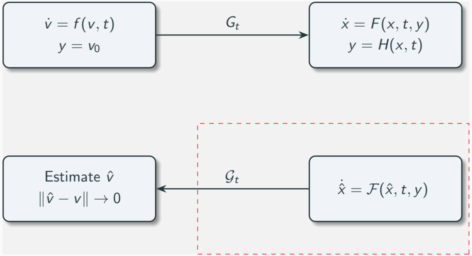



## Design approach

$$\begin{cases} \text{If } v(t) \in \mathcal{T}: \\ \dot{\hat{v}}(t) = f(\hat{v}, t), \ \hat{x}(t) = G_t(\hat{v}(t)) \\ \\ \text{If } v(t) \notin \mathcal{T}: \\ \hat{v}(t) = \mathcal{G}_t(\hat{x}(t)), \quad \dot{\hat{x}} = \mathcal{F}(\hat{x}, t, y) \end{cases}$$



# Embedding

$$\begin{cases}\dot v = f(v,t)\\ y = v_0\end{cases} \xrightarrow{\;G_t\;} \begin{cases}\dot x = F(x,t,y)\\ y = x_0\end{cases}$$

## Embedding in a triangular form

Assume $|\det(I_{1:2}, \dot I_{1:2})| > \mu > 0$ and $I_0 > c > 0$. For all $t \geq 0$, there exist an explicit **injective** map $G_t : \mathbb{R}^3 \setminus \mathcal{T} \to \mathbb{R}^4$ that transform system $(\star)$ into a triangular form

$$\begin{cases}\dot x_0 = x_1 + \phi_0(x_0, t)\\ \dot x_1 = x_2 + \phi_1(x_0, x_1, t)\\ \dot x_2 = x_3 + \phi_2(x_0, x_1, x_2, t)\\ \dot x_3 = \phi_3(x_0, x_1, x_2, x_3, t)\\ y = x_0\end{cases}$$

# Triangular Form

$$\begin{cases} \dot v_0 = -v_0 + w_0 \Gamma_0^0(v_0, |v_{1:2}|) + I_0 \\ \dot v_{1:2} = -v_{1:2} + w_1 \Gamma_0^1(v_0, |v_{1:2}|) \frac{v_{1:2}}{|v_{1:2}|} + I_{1:2} \end{cases}$$

$$\begin{cases} \dot x_0 = x_1 + \phi_0(x_0, t) \\ \dot x_1 = x_2 + \phi_1(x_0, x_1, t) \\ \dot x_2 = x_3 + \phi_2(x_0, x_1, x_2, t) \\ \dot x_3 = \phi_3(x_0, x_1, x_2, x_3, t) \end{cases}$$

**The non-linearities $\phi_i$ are not defined everywhere**

**Hölder continuity of the non-linearities**

The non-linearities $\phi_i$ can be extended to be defined everywhere in $\mathbb{R}^4$. Moreover, there exist $\bar b > 0$ and $0 \leq \alpha_{ij} < 1$ such that for all $x$, $\tilde x$ in $\mathbb{R}^4$

$$|\phi_i(x_0, .., x_j, t) - \phi_i(\tilde x_0, .., \tilde x_j, t)| \leq \bar b \sum_{j=0}^{i} |x_j - \tilde x_j|^{\alpha_{ij}}$$

Obtained through analysis of $\Gamma_i^j(v_0, \rho) := \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{+\infty} S^{(i)}\big(v_0 + \rho r \cos 2\theta\big)(r \cos 2\theta)^j \frac{P(r)}{\pi} \mathrm{d}r\mathrm{d}\theta$

# Triangular Form

## Hölder continuity of the non-linearities

There exist $\bar{b} > 0$ and $0 \leq \alpha_{ij} < 1$ such that for all $x$, $\tilde{x}$ in $\mathbb{R}^4$

$$|\phi_i(x_0, .., x_j, t) - \phi_i(\tilde{x}_0, .., \tilde{x}_j, t)| \leq \bar{b} \sum_{j=0}^{i} |x_j - \tilde{x}_j|^{\alpha_{ij}}$$

$\alpha_{ij} \geq$

| $i \backslash j$ | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| 0 | $\frac{3}{4}$ | | | |
| 1 | $\frac{1}{2}$ | $\frac{2}{3}$ | | |
| 2 | $\frac{1}{4}$ | $\frac{1}{3}$ | $\frac{1}{2}$ | |
| 3 | 0 | 0 | 0 | 0 |

## Observer in the embeded domain

It is possible to design a sliding mode observer for the triangular form, ensuring finite-time convergence[1] [2]

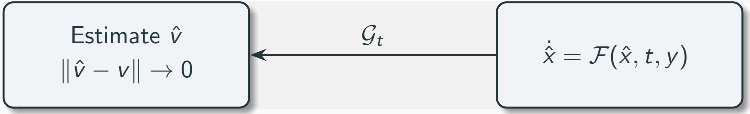
[1] P. Bernard, L. Praly, V. Andrieu (2017)
[2] V. Andrieu, L. Praly, and A. Astolfi. (2008)

# Pseudo Inverse


Usual method for inversion

$$\hat{v} = \arg\min\{|G_t(v) - \hat{x}|\}$$

- **Computationally costly**
- **Local minima**




## Pseudo Inverse

Assume $|\det(l_{1:2}, \dot{l}_{1:2})| > \mu > 0$ and $l_0 > c > 0$. There exist an explicit map $\mathcal{G} : \mathbb{R}^4 \times \mathbb{R} \to \mathbb{R}^3$ and a class $\mathcal{K}$ function $\omega$ such that for every time $t \geq 0$

$$\mathcal{G}_t\left(G_t(v)\right) = v, \;\; v \in \mathbb{R}^3 \setminus \mathcal{T}$$

$$|\mathcal{G}_t(x) - \mathcal{G}_t(\tilde{x})| \leq \omega(|x - \tilde{x}|), \quad (x, \tilde{x}) \in \mathbb{R}^4 \times \mathbb{R}^4$$

# Observer Design

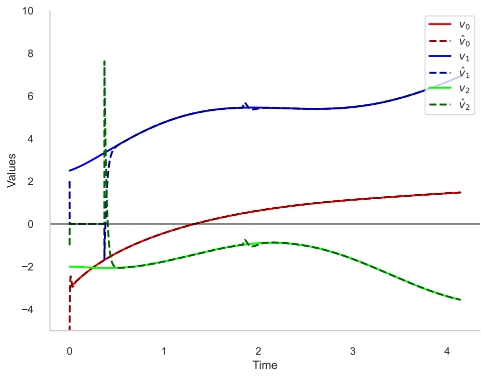
$$\begin{cases} \text{If } v(t) \in \mathcal{T}: \\ \dot{\hat{v}}(t) = f(\hat{v}, t),\ \hat{x}(t) = G_t(\hat{v}(t)) \\ \text{If } v(t) \notin \mathcal{T}: \\ \hat{v}(t) = \mathcal{G}_t(\hat{x}(t)), \quad \begin{cases} \dot{\hat{x}}_0 = \hat{\phi}_0 + \hat{x}_1 - Lk_0\lceil \hat{x}_0 - y \rfloor^{\frac{3}{4}} \\ \dot{\hat{x}}_1 = \hat{\phi}_1 + \hat{x}_2 - L^2 k_1\lceil \hat{x}_0 - y \rfloor^{\frac{1}{2}} \\ \dot{\hat{x}}_2 = \hat{\phi}_2 + \hat{x}_3 - L^3 k_2\lceil \hat{x}_0 - y \rfloor^{\frac{1}{4}} \\ \dot{\hat{x}}_3 \in \hat{\phi}_3 - L^4 k_3 \mathrm{sign}(\hat{x}_0 - y) \end{cases} \end{cases}$$

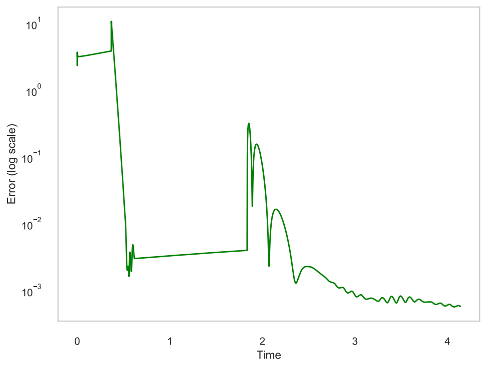
**Theorem**

Assume $|\det(l_{1:2}, \dot{l}_{1:2})| > \mu > 0$ and $l_0 > c > 0$. Let $R > 0$ be large enough. There exist gains $k_i > 0$ and $L^* > 0$ such that for every $L \geq L^*$ there exists a class $\mathcal{KL}$ function $\mathcal{B}$ such that for every solutions of the above observer initialised with $\hat{v}(0) \in B_{\mathbb{R}^3}(0, R)$, $\hat{z}(0) = G_0(\hat{v}(0))$ we have

$$|v(t) - \hat{v}(t)| \leq \mathcal{B}(|v(0) - \hat{v}(0)|, t), \quad t \geq 0$$

# Numerical simulations



# Conclusion

- We characterized a necessary and sufficient condition for the observability of system $(\star)$
- We designed a hybrid observer that exhibits finite time convergence
- The observability and observer design study is due to the nonlinear nature of the model
- The "embedded observer" can be modified as long as it copes with the Hölder nature of the non linearities
- Stability discussion (CDC Paper)

# Thank You!

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