Microscopic locomotion and controllability of affine systems

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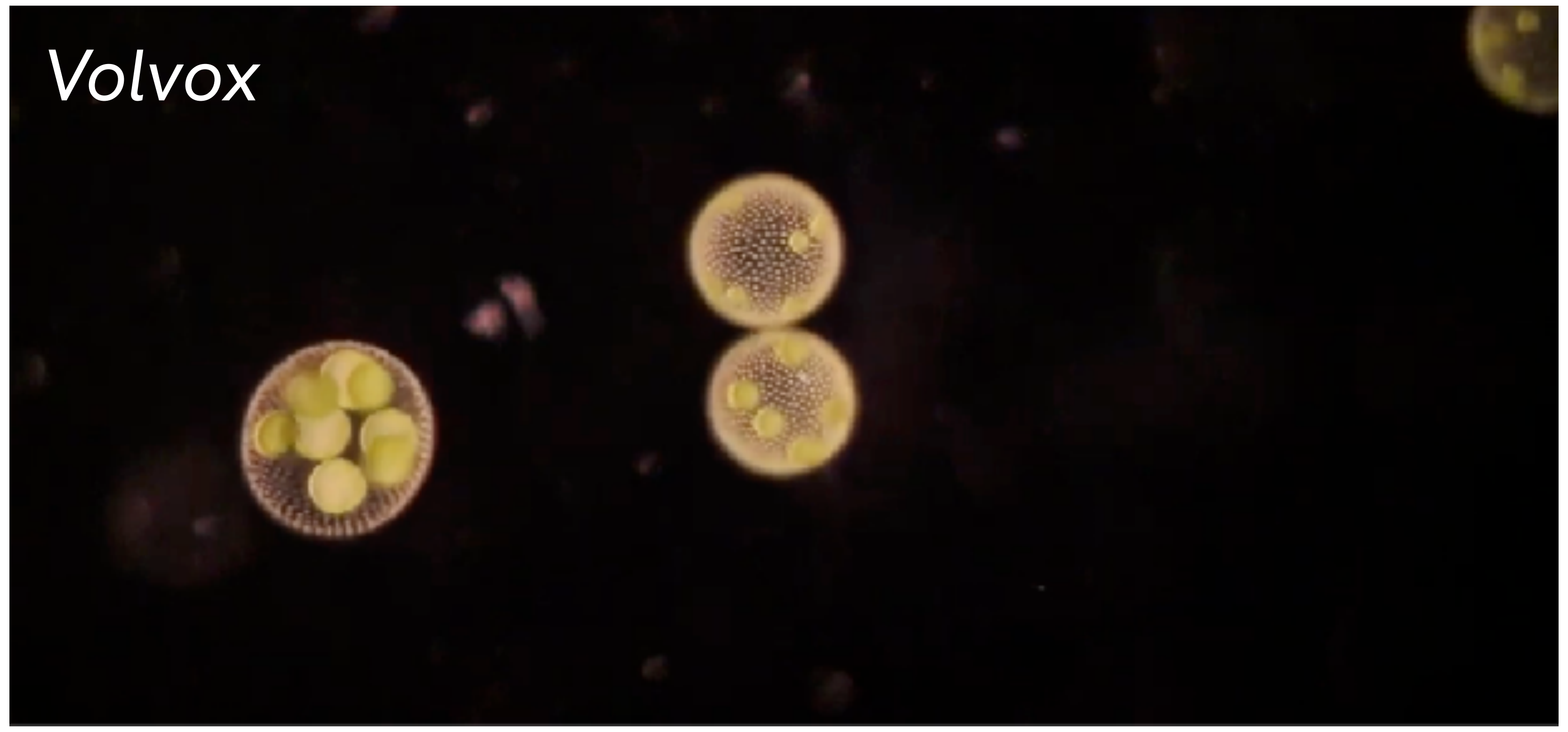
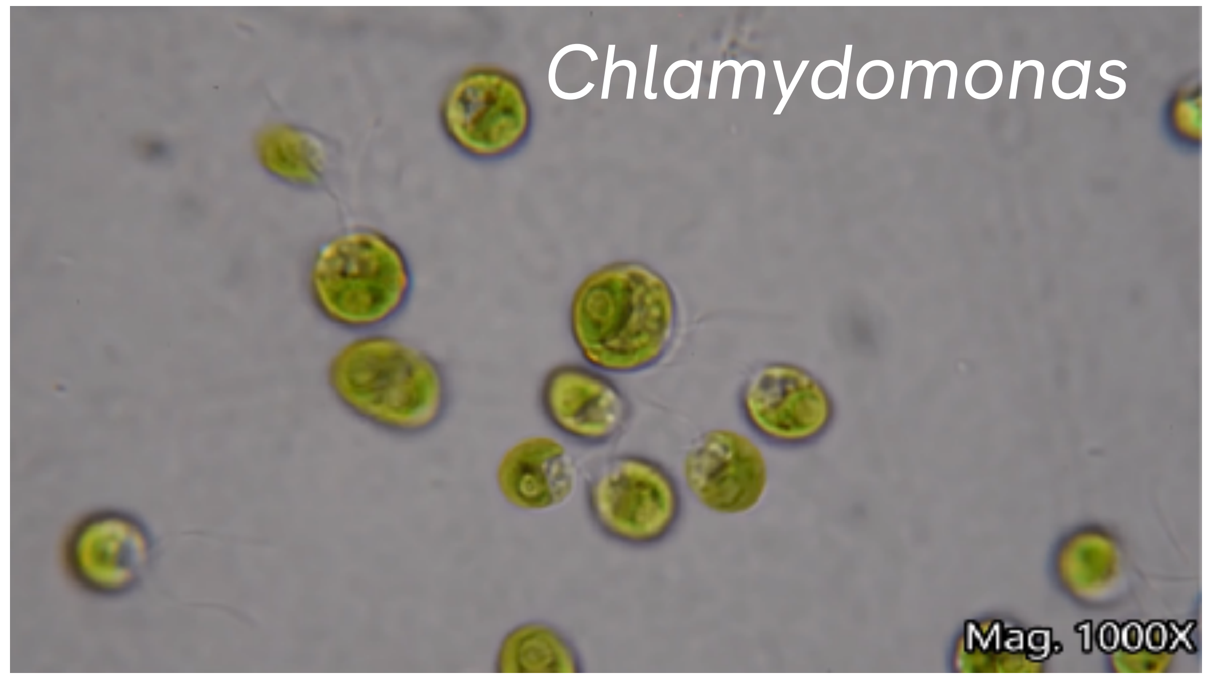
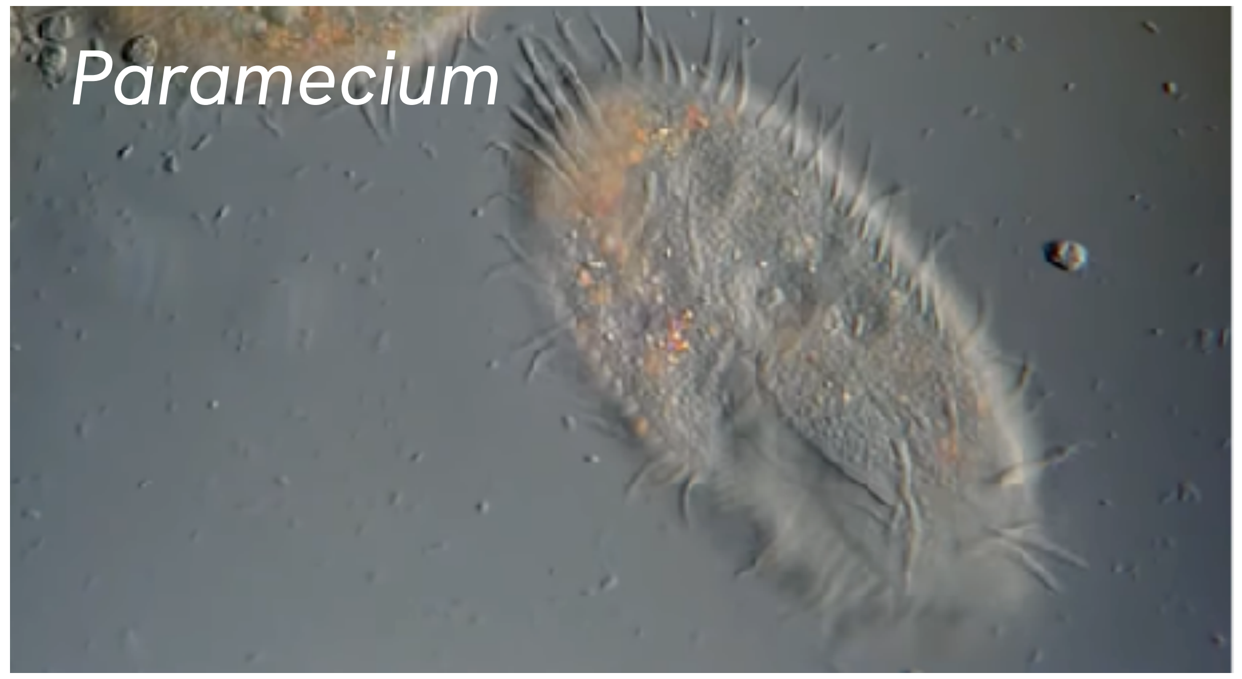
Laboratoire des Sciences du Numérique de Nantes

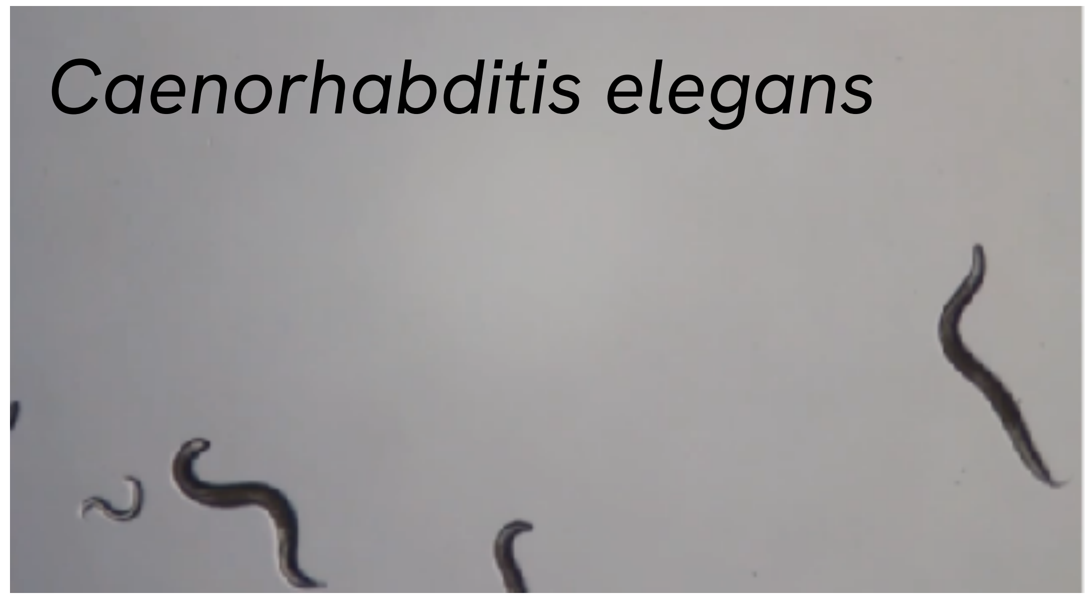
Séminaire d'automatique du plateau de Saclay — November 4th, 2025

Microscopic locomotors



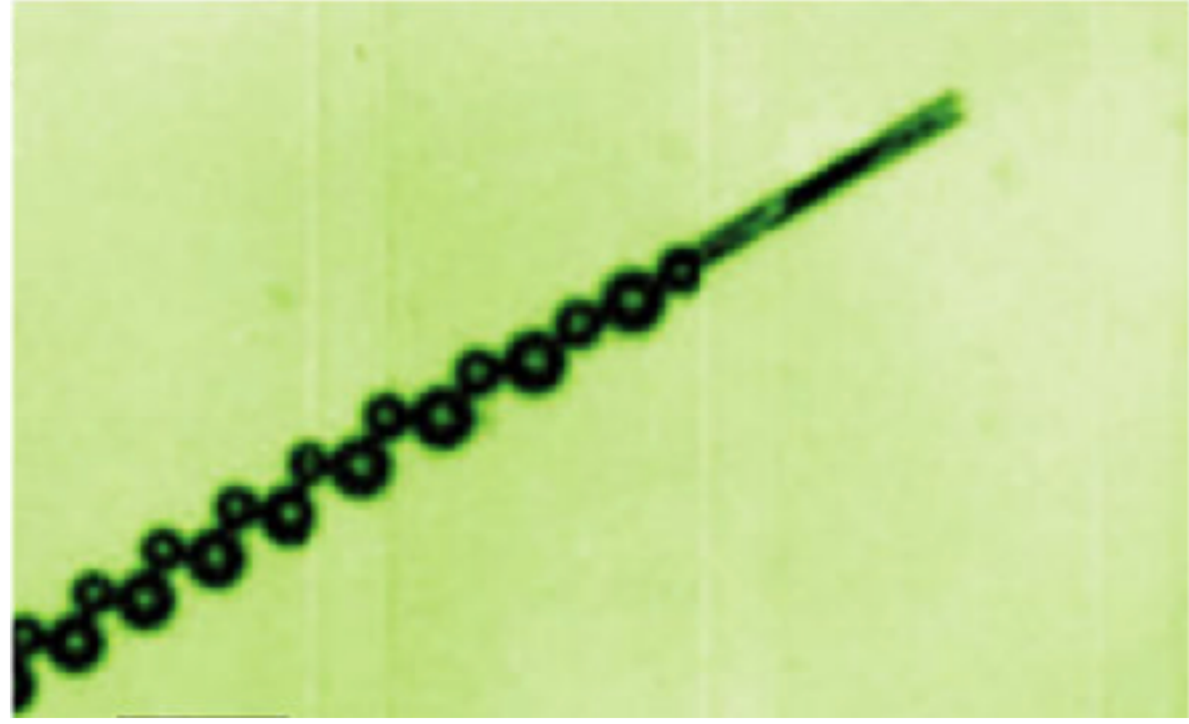




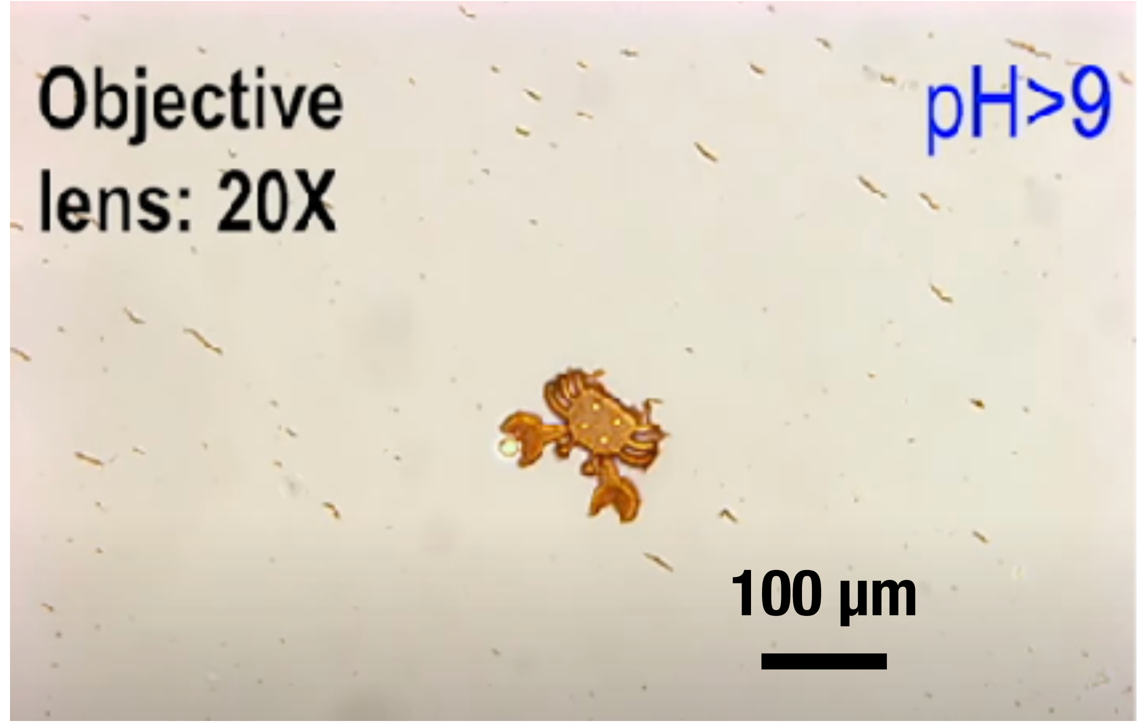


Micro-robots

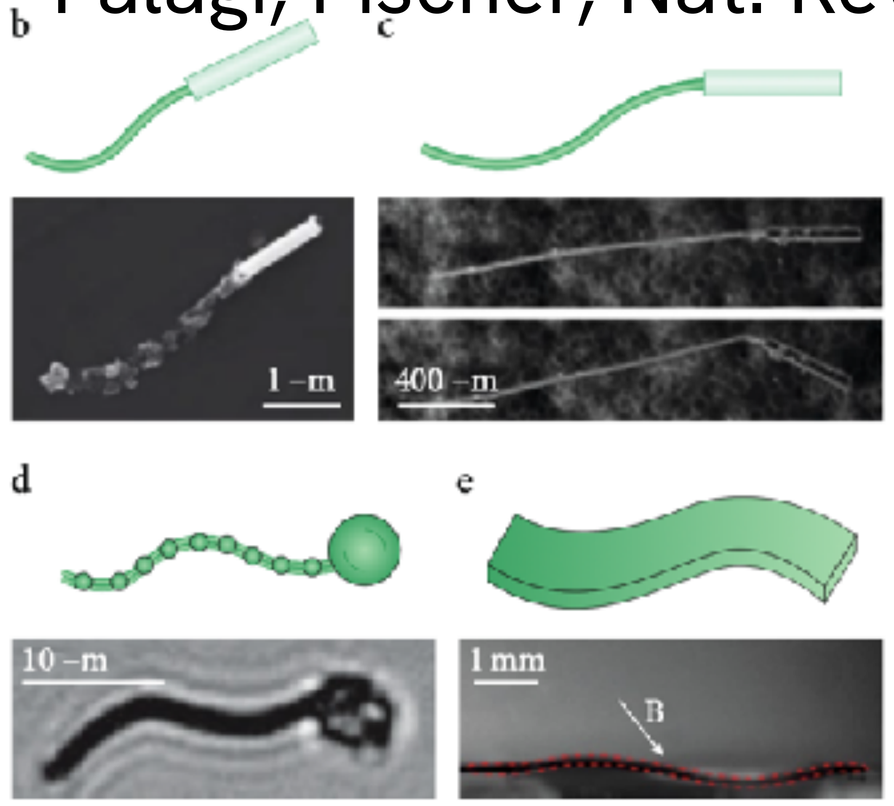
Sovolev et al., Small 5(14), 2009

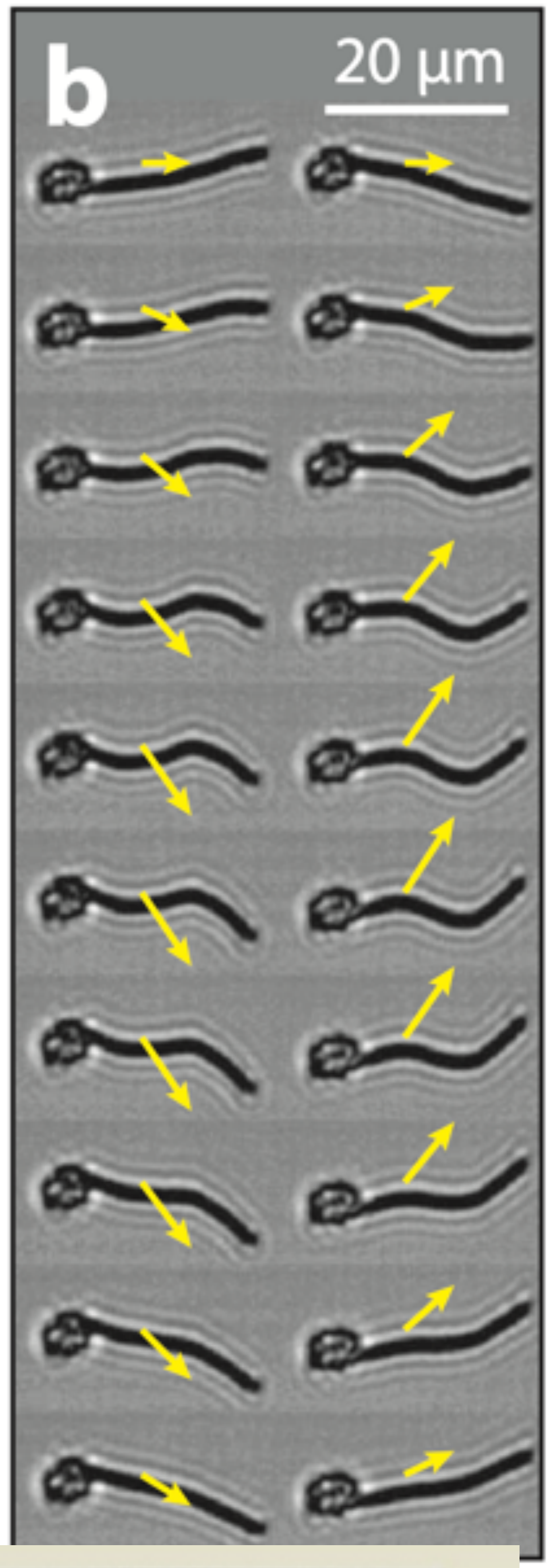


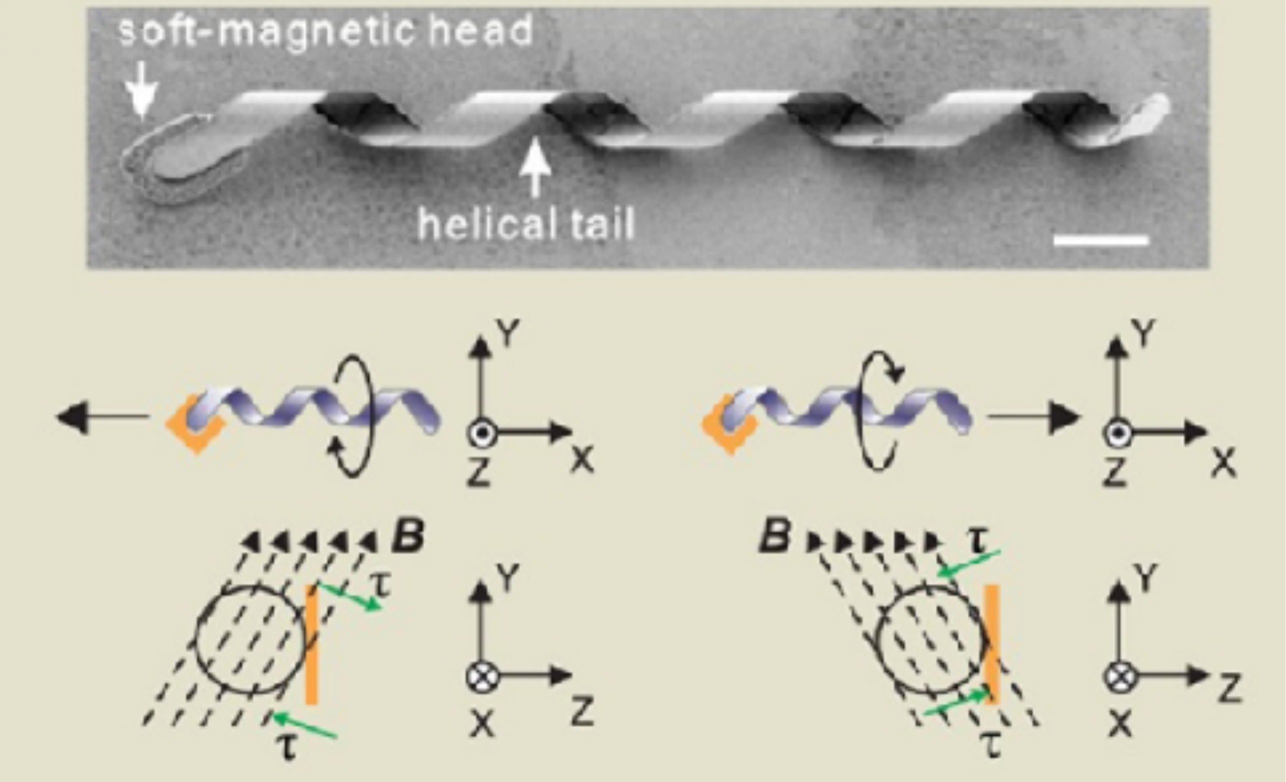
Xin et al., ACS Nano 15(11), 2021

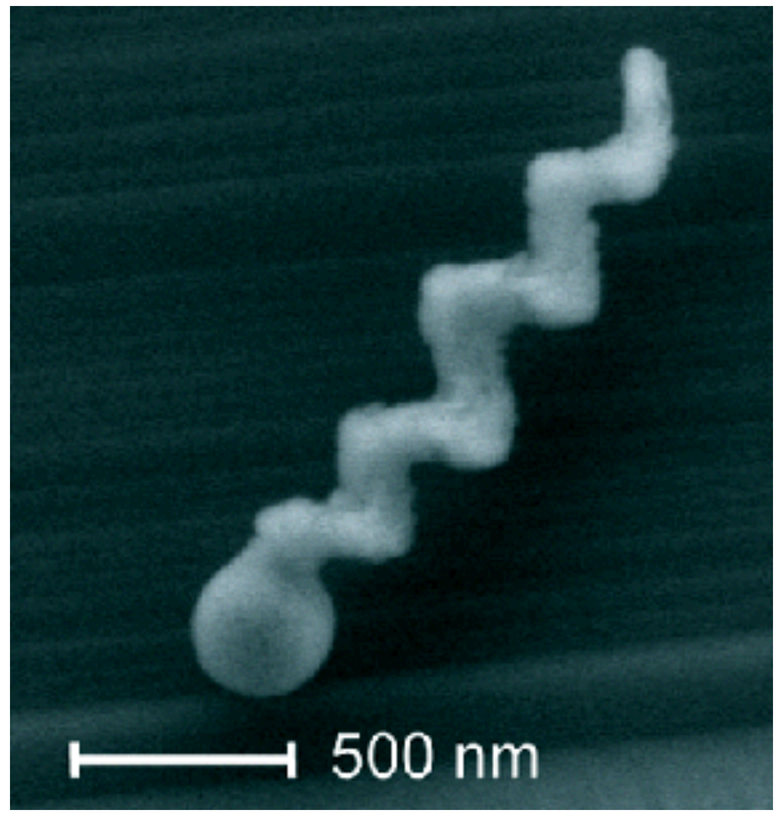


Palagi, Fischer, Nat. Rev. Materials 3 (2018)

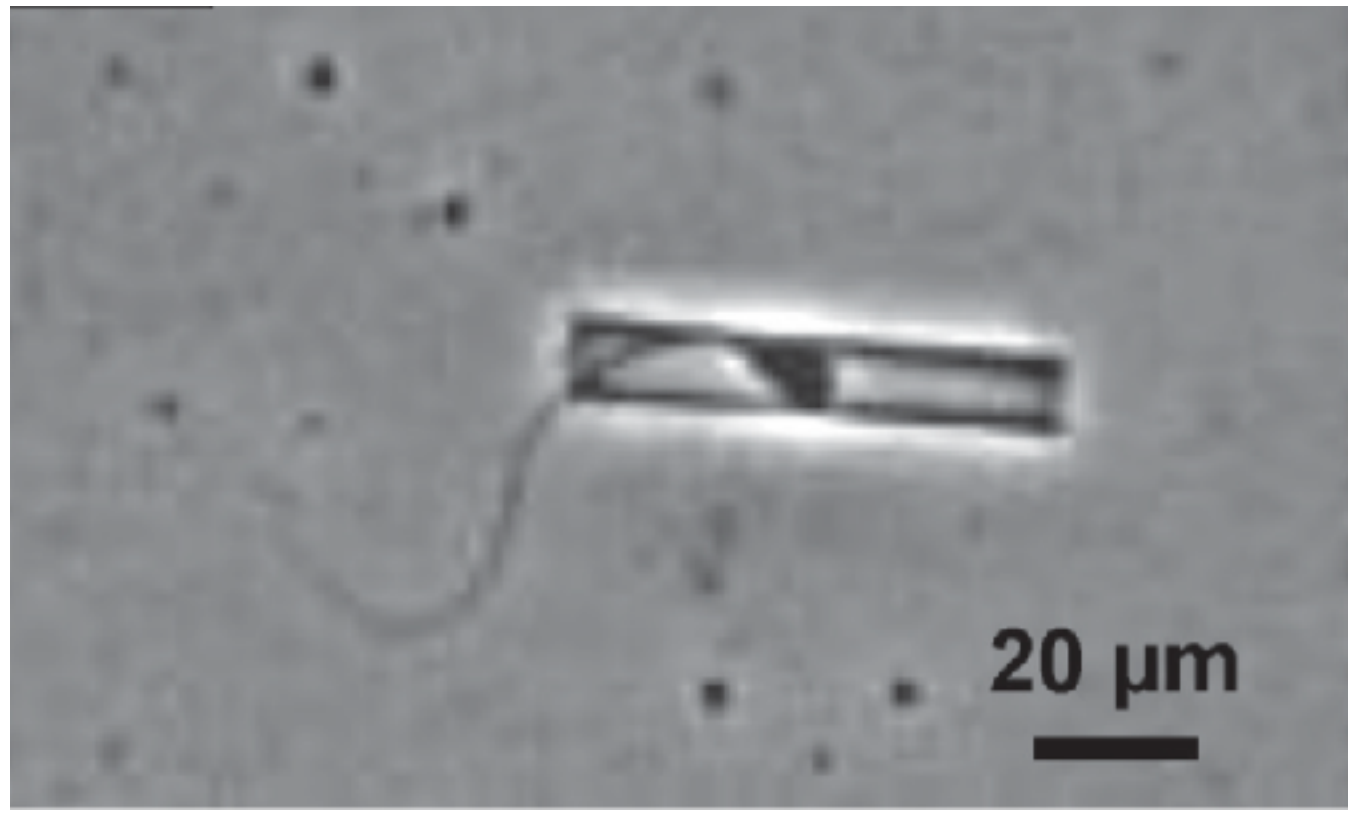




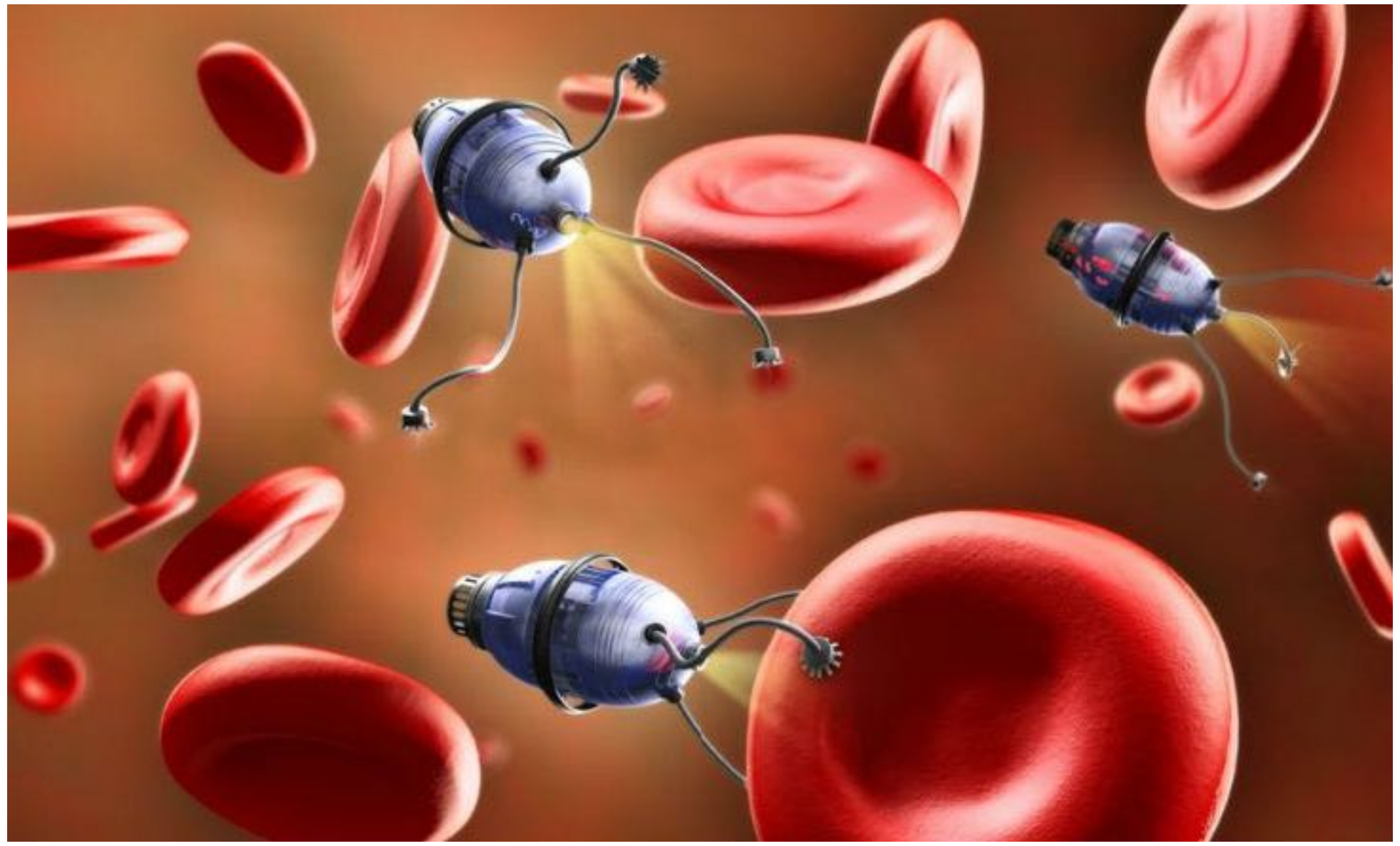


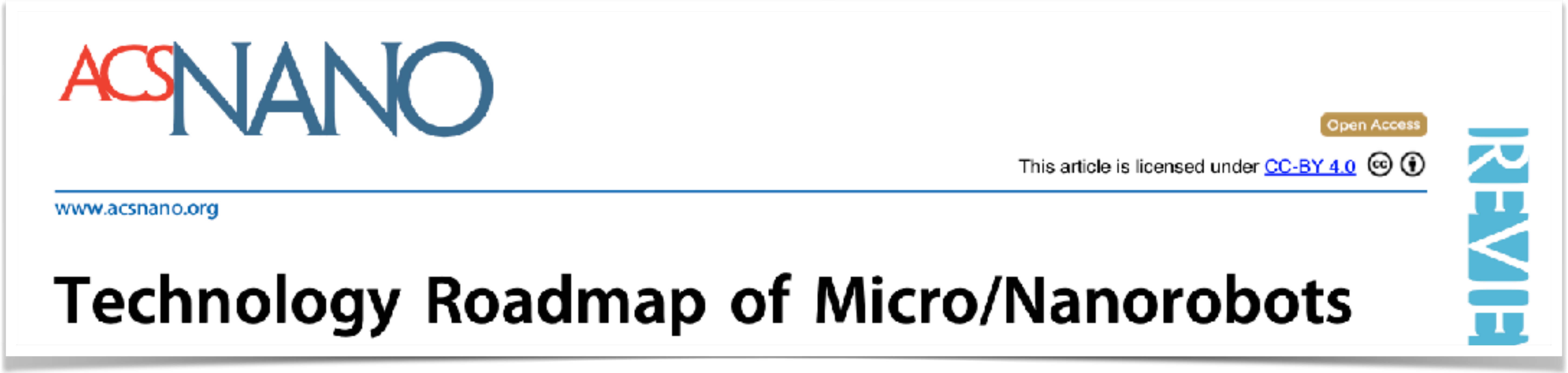


Ghosh & Fischer, Nano Letters 9(6), 2009



Magdanz et al., Adv. Funct. Mater. 25(18), 2015

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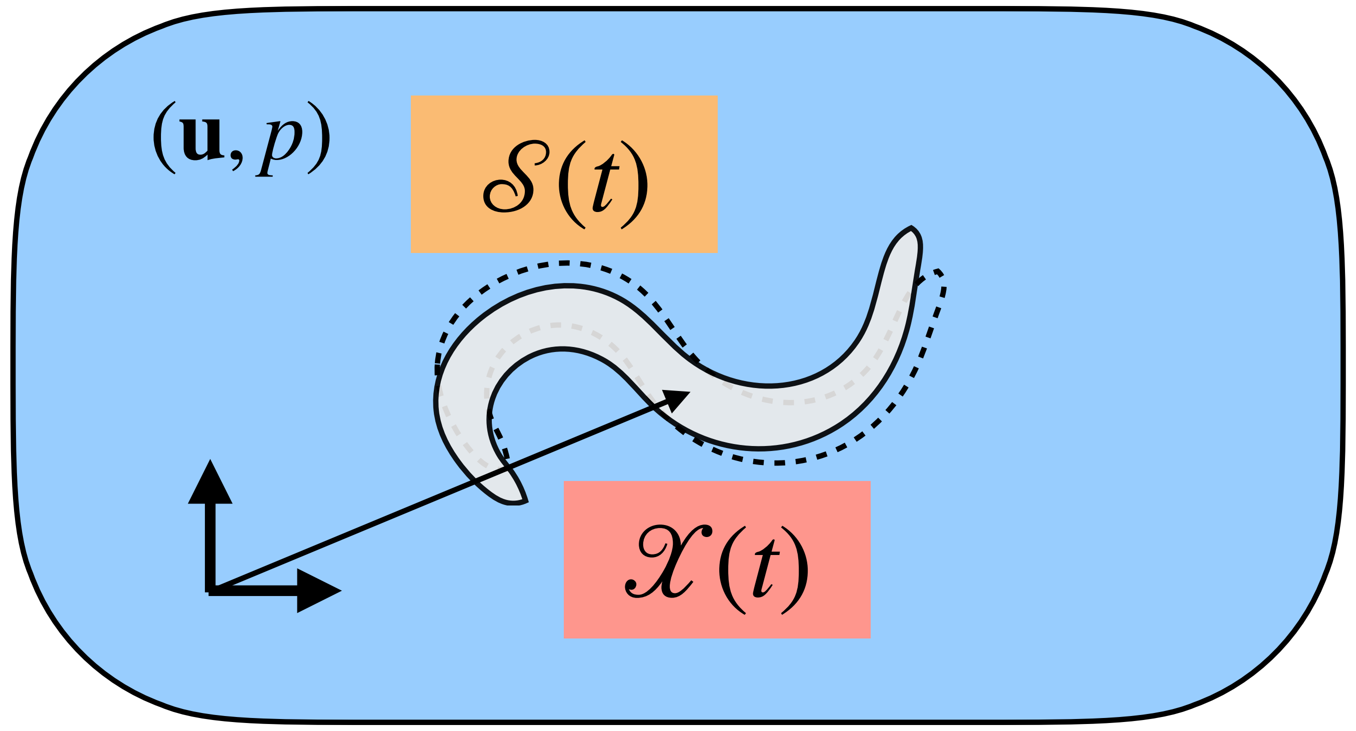
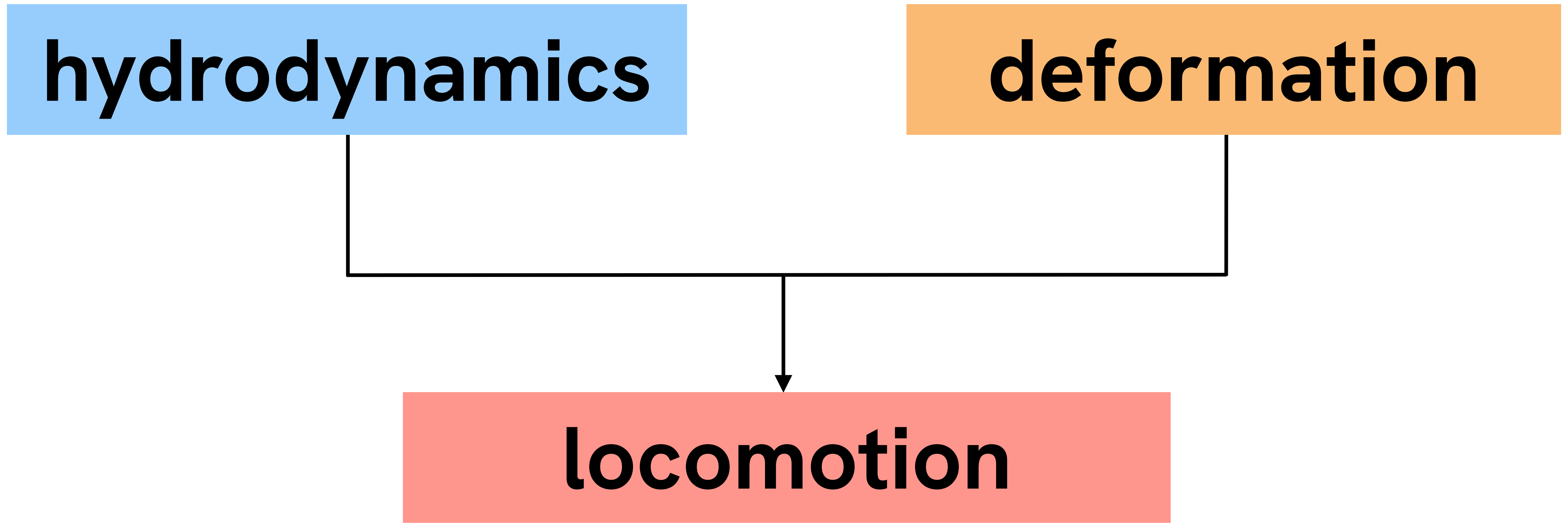
Technology Roadmap of Micro/Nanorobots

Principles of micro-swimming





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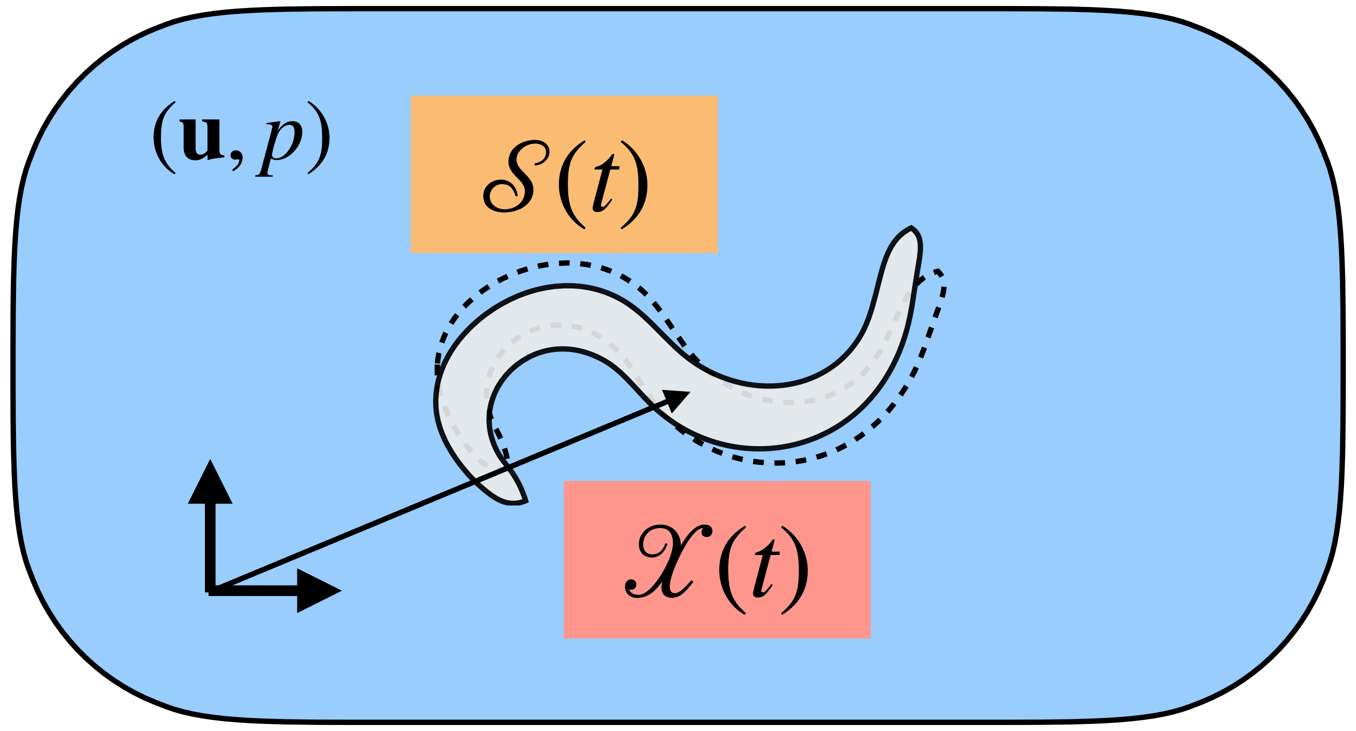


hydrodynamics

deformation

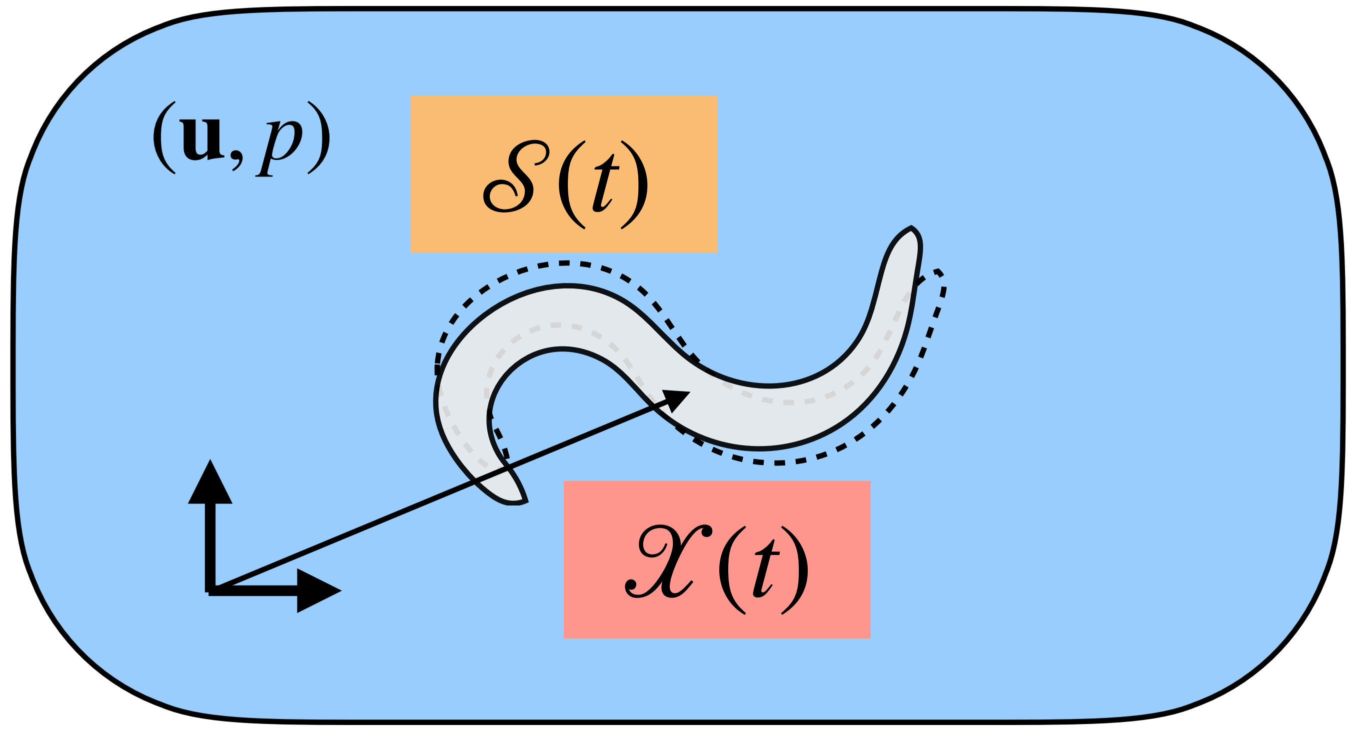
locomotion

rigid motion

$\mathcal{X}(t)$

Pose in $SO(n)$

Principles of micro-swimming



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rate of deformation $\dot{\mathcal{S}}(t)$ « Degrees of freedom », finite (or infinite) dimension

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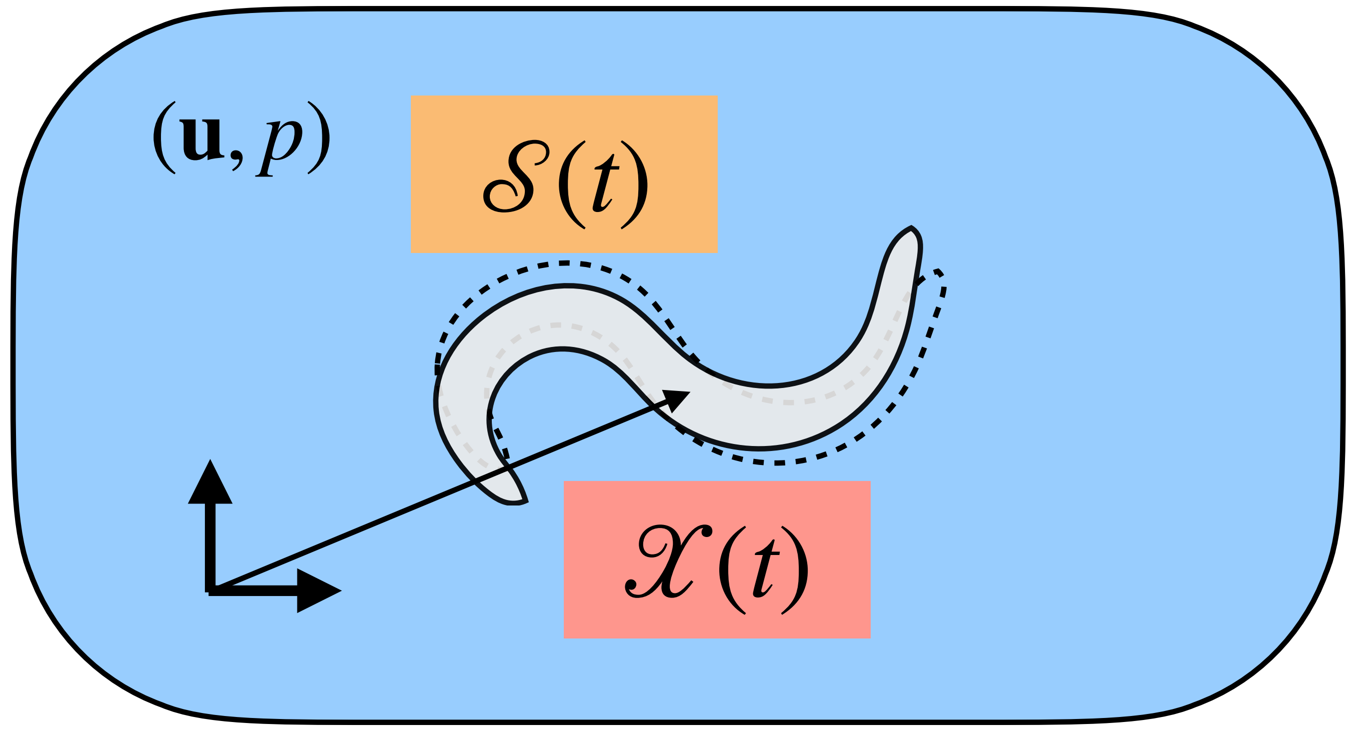

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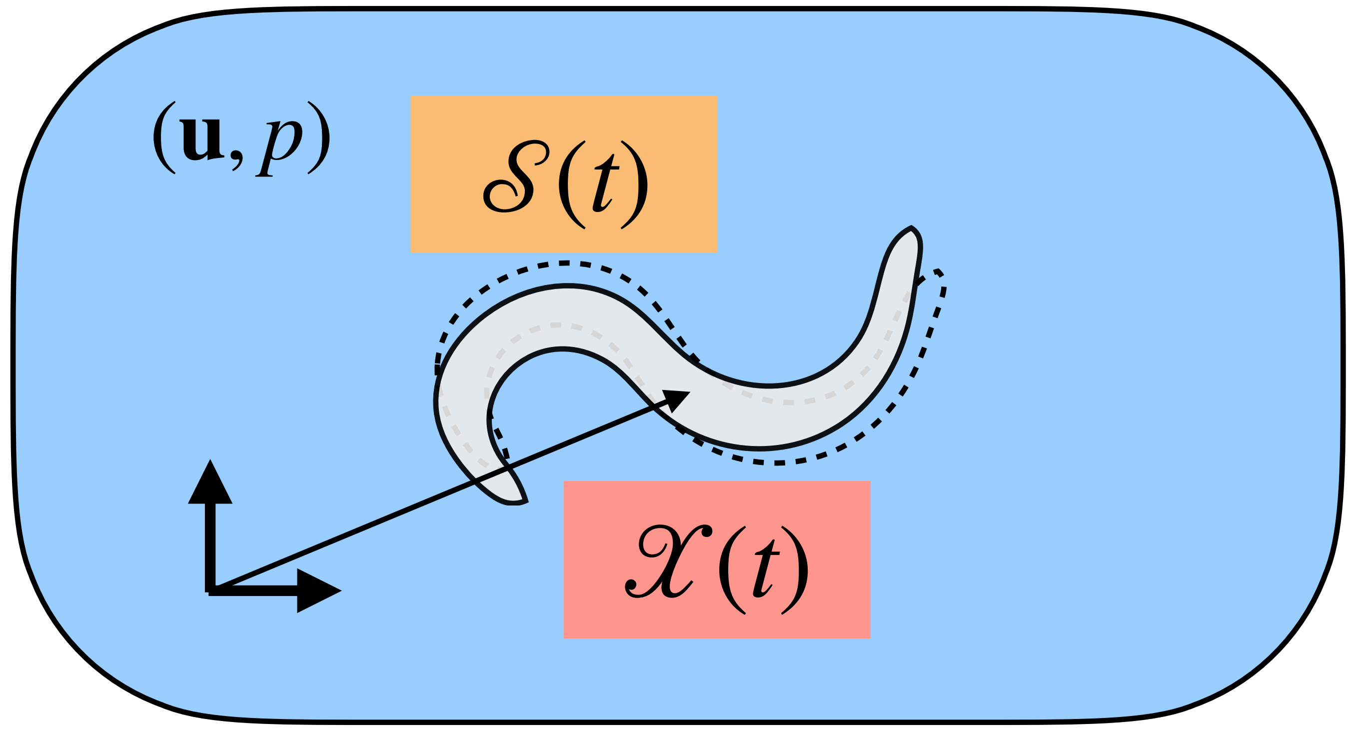
Stokes flow

(Zero Reynolds number)

$$\Delta \mathbf{u} + \nabla p = 0, \quad \nabla \cdot u = 0$$

B.C. : $\dot{\mathscr{X}} + \dot{\mathscr{S}}$

Principles of micro-swimming



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Stokes flow

- Linearity:

$$\mathbf{F}^{\text{hydro}} = \mathbf{Q}(\mathcal{S})(\dot{\mathcal{X}}, \dot{\mathcal{S}})^T$$

Stokes flow

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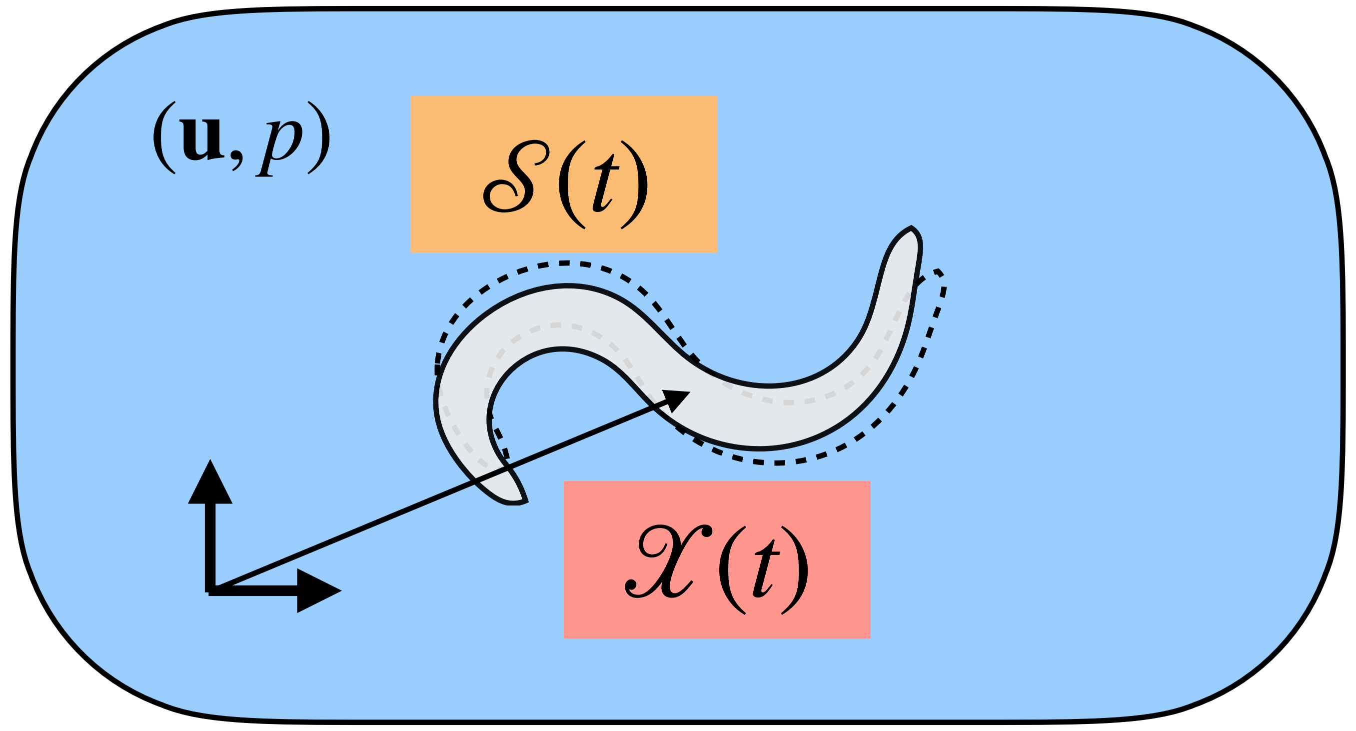
Principles of micro-swimming



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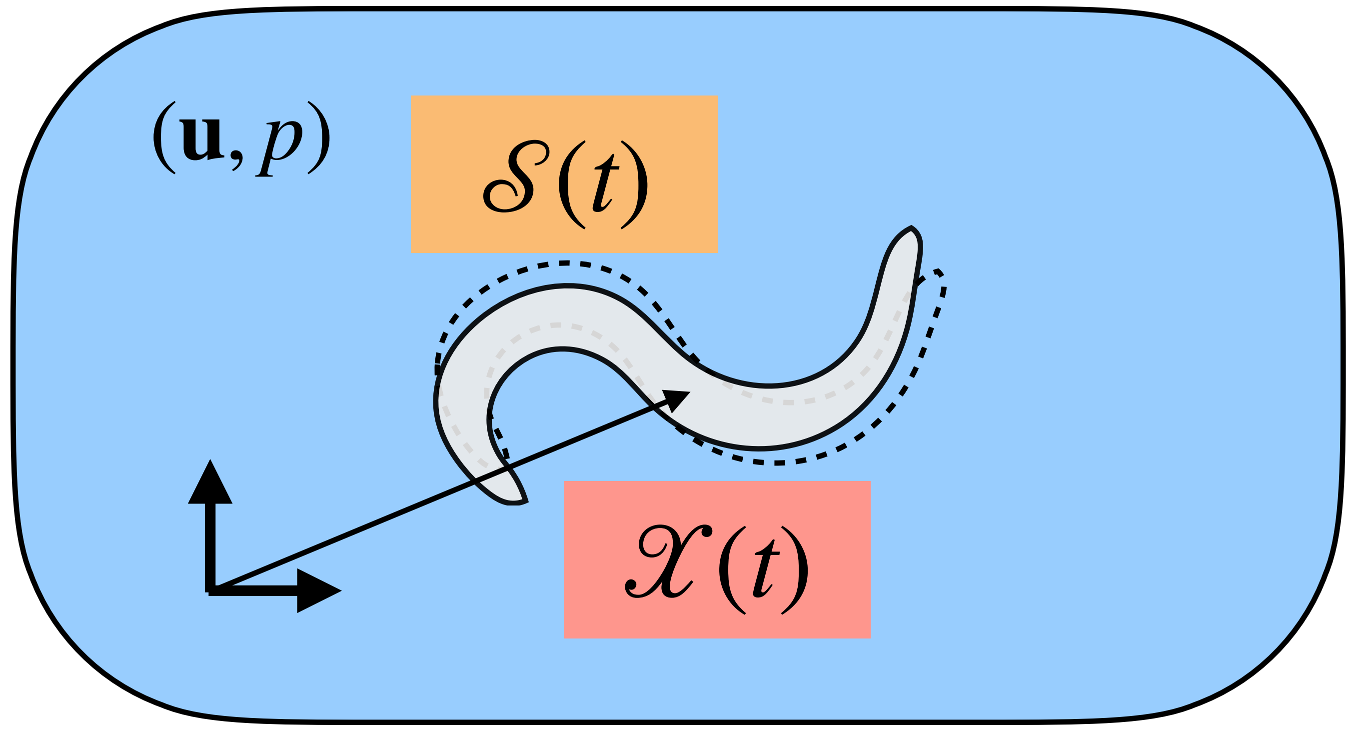

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$$\mathbf{F}^{\text{hydro}} = 0$$

→

« Stokes connection »

$$\dot{\mathcal{X}} = \mathbf{N}(\mathcal{S})\dot{\mathcal{S}}$$

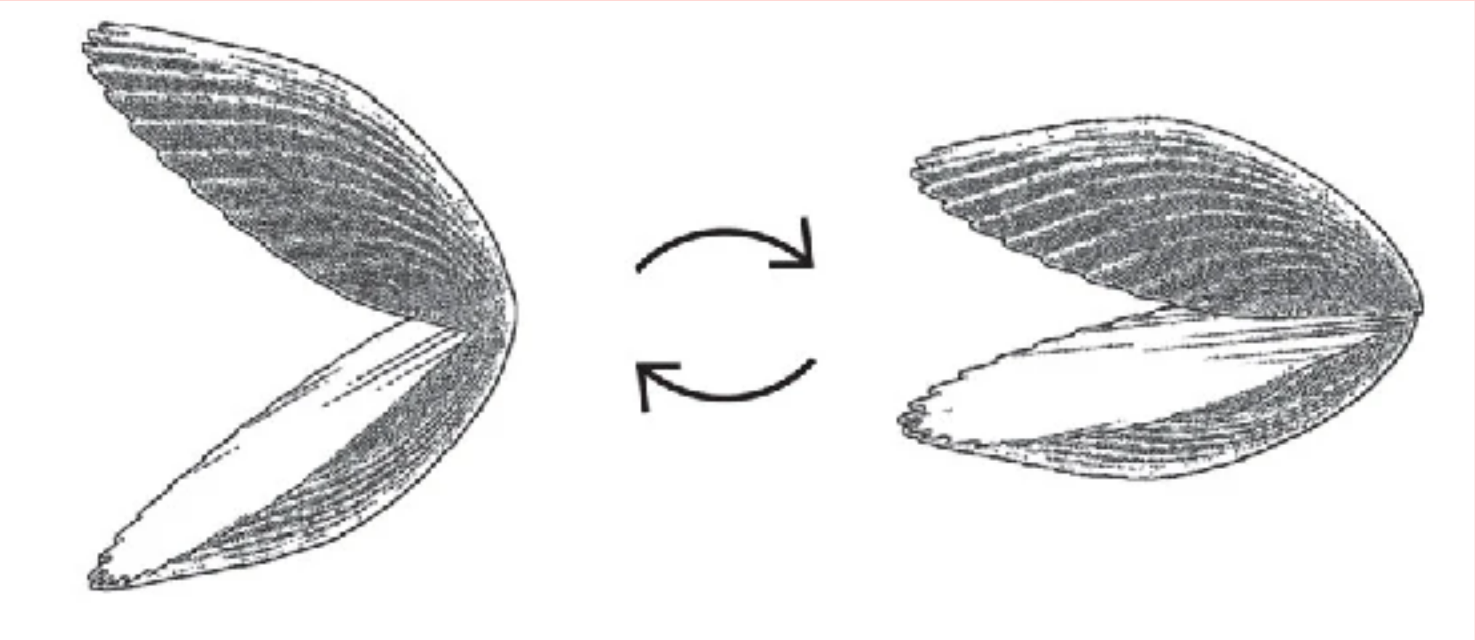
Swimming equation

Principles of microswimming

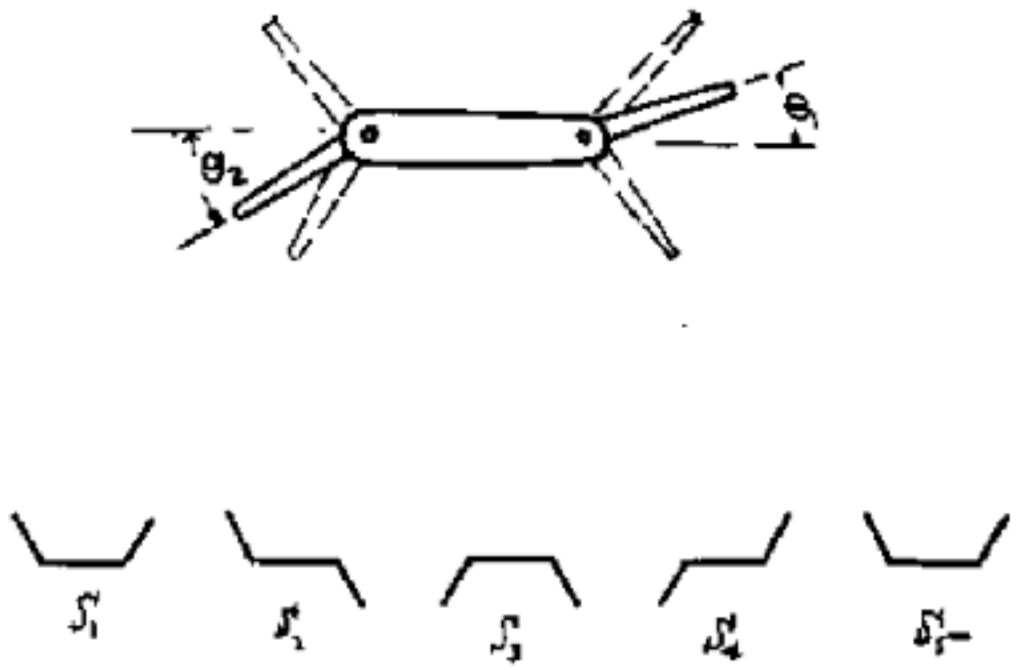
Consequence: Scallop theorem

"A swimmer with only one degree of freedom cannot produce net locomotion at low Reynold's number"

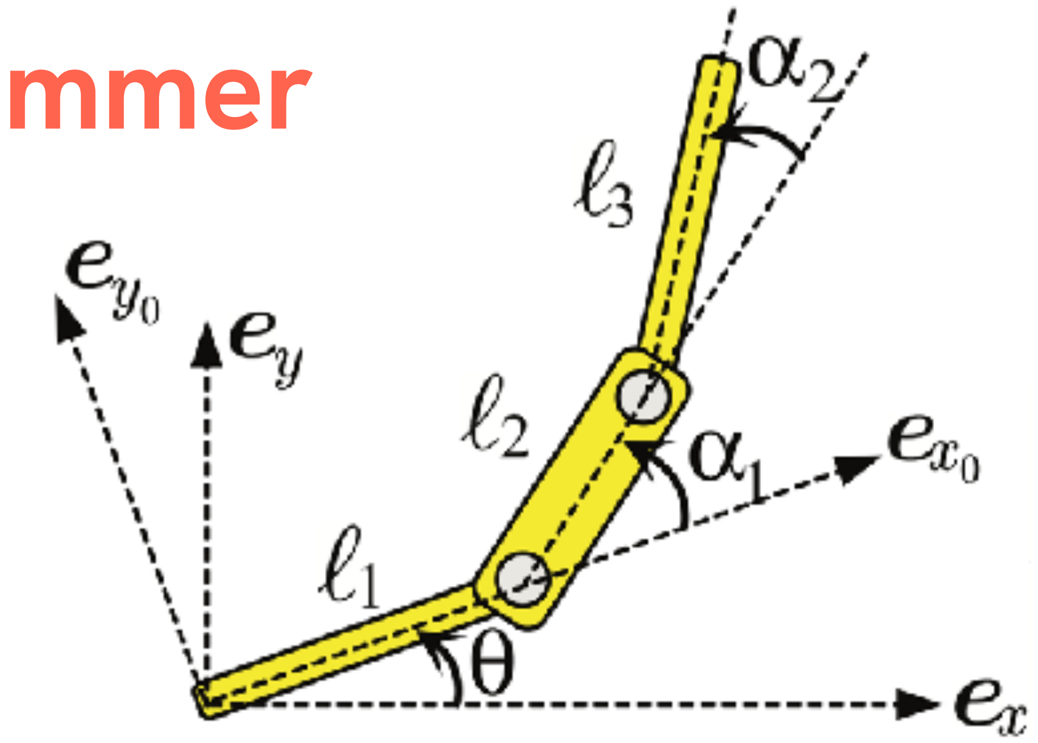
→ **Non-reciprocal** deformation is required



- **The famous Purcell swimmer**

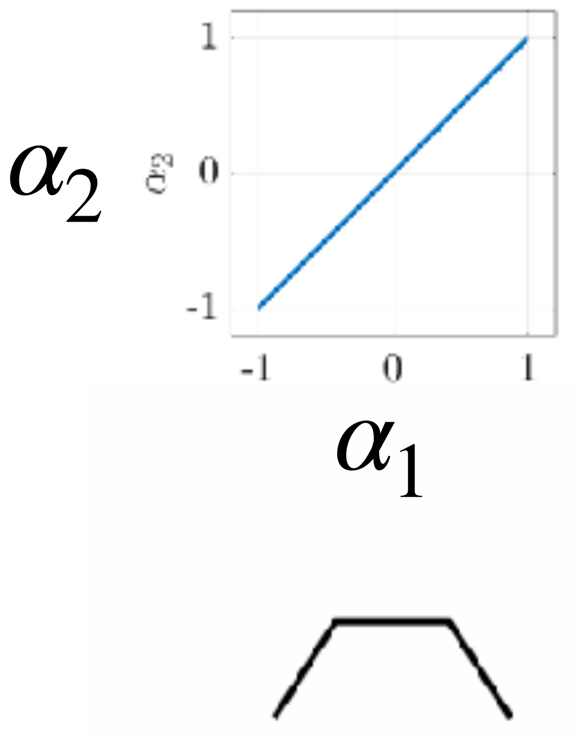


Purcell, Am. J. Phys., 1977



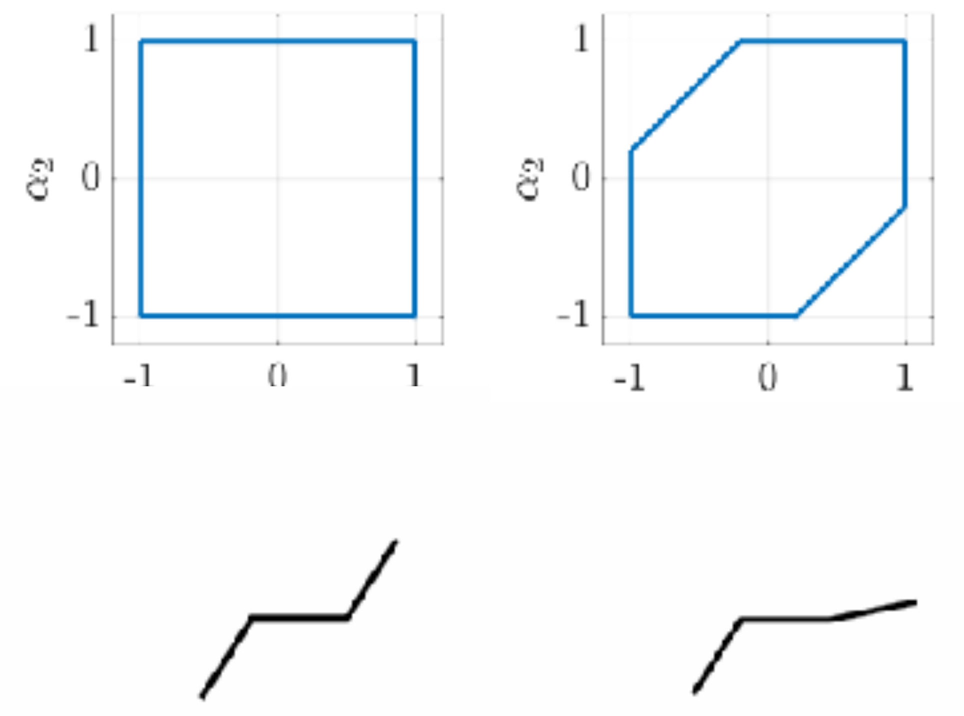
Three rigid links with flexible junctions

Reciprocal gait



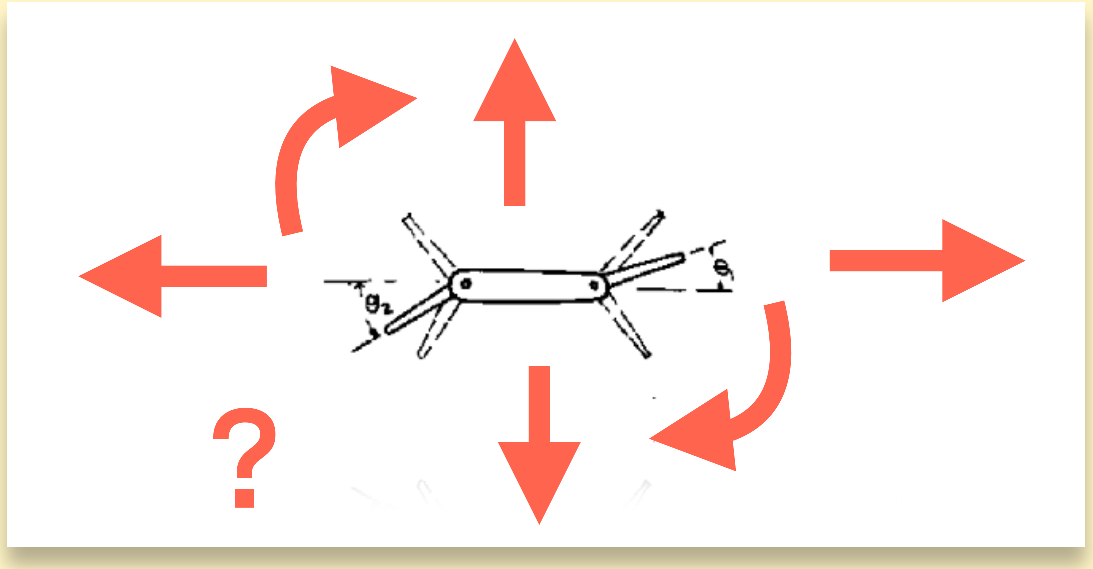
No net motion

Non-reciprocal gait



Net motion

“Non-reciprocal deformation is required for net locomotion”



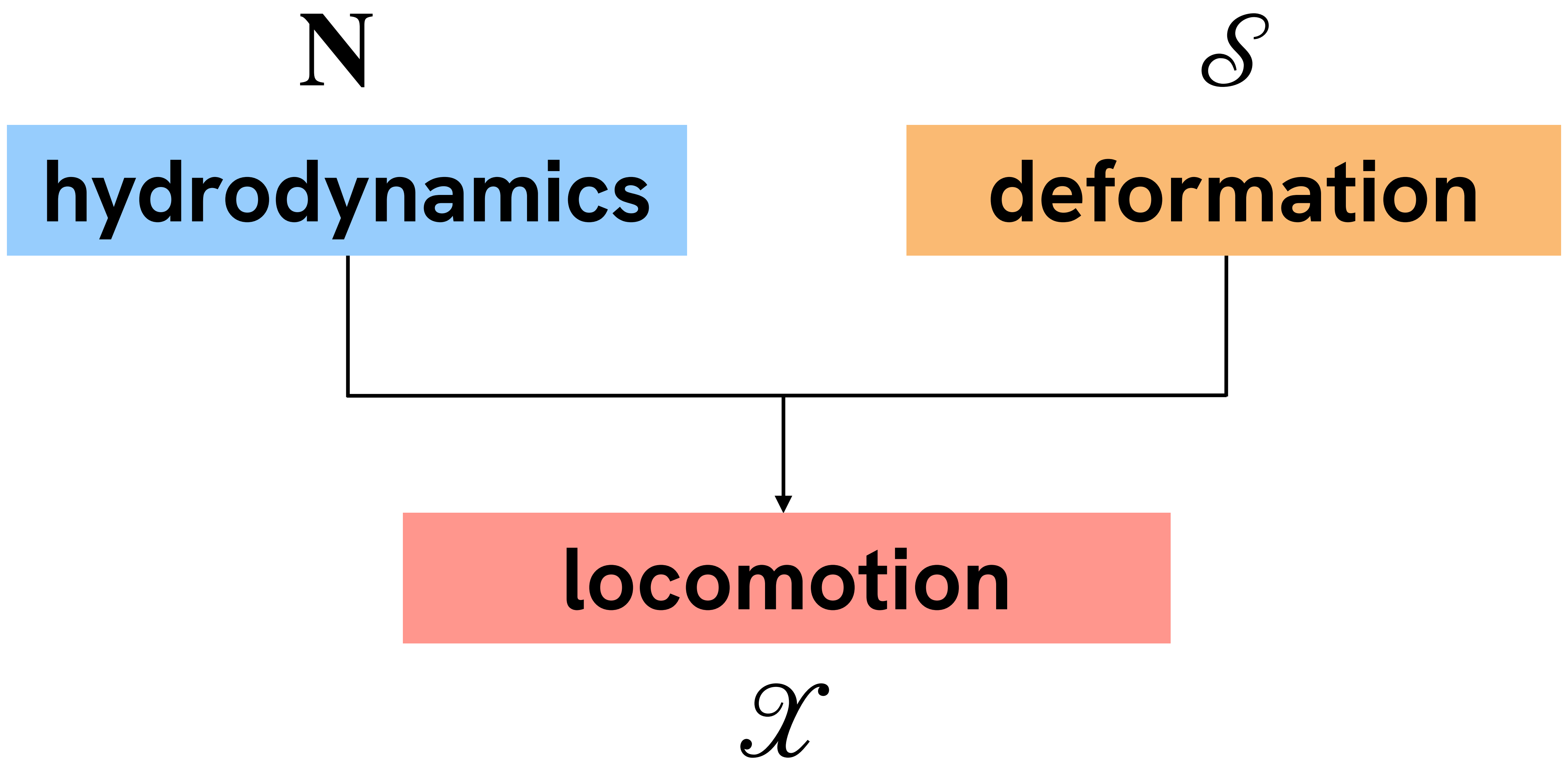
Question:

Which kind of net locomotion?

Is **every direction** reachable as soon as I can do a “nonreciprocal” deformation?

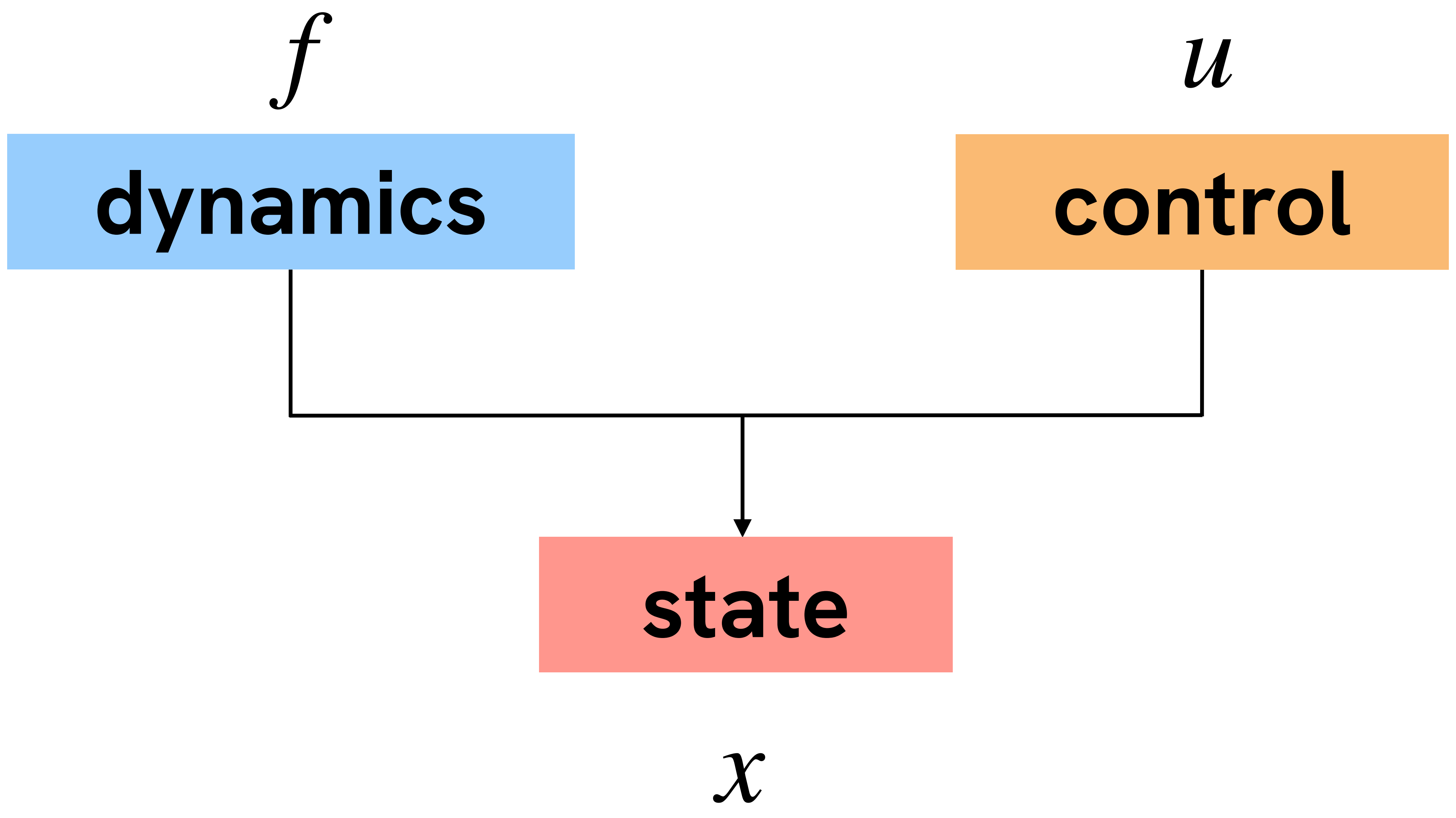
→ **controllability**

Swimming is a driftless control-affine system



$$\dot{\mathcal{X}} = \mathbf{N}(\mathcal{S})\dot{\mathcal{S}}$$

Swimming equation



$$\dot{x} = \sum_{i=1}^{m} f_i(x)u_i$$

Control-affine system

Swimming is a driftless control-affine system

$\mathbf{N}$ hydrodynamics

$\mathcal{S}$ deformation

→ locomotion $\mathcal{X}$

$$\begin{pmatrix} \dot{\mathcal{X}} \\ \dot{\mathcal{S}} \end{pmatrix} = \begin{pmatrix} \mathbf{N}(\mathcal{S}) \\ \mathrm{Id} \end{pmatrix} \dot{\mathcal{S}}$$

Swimming equation

f dynamics

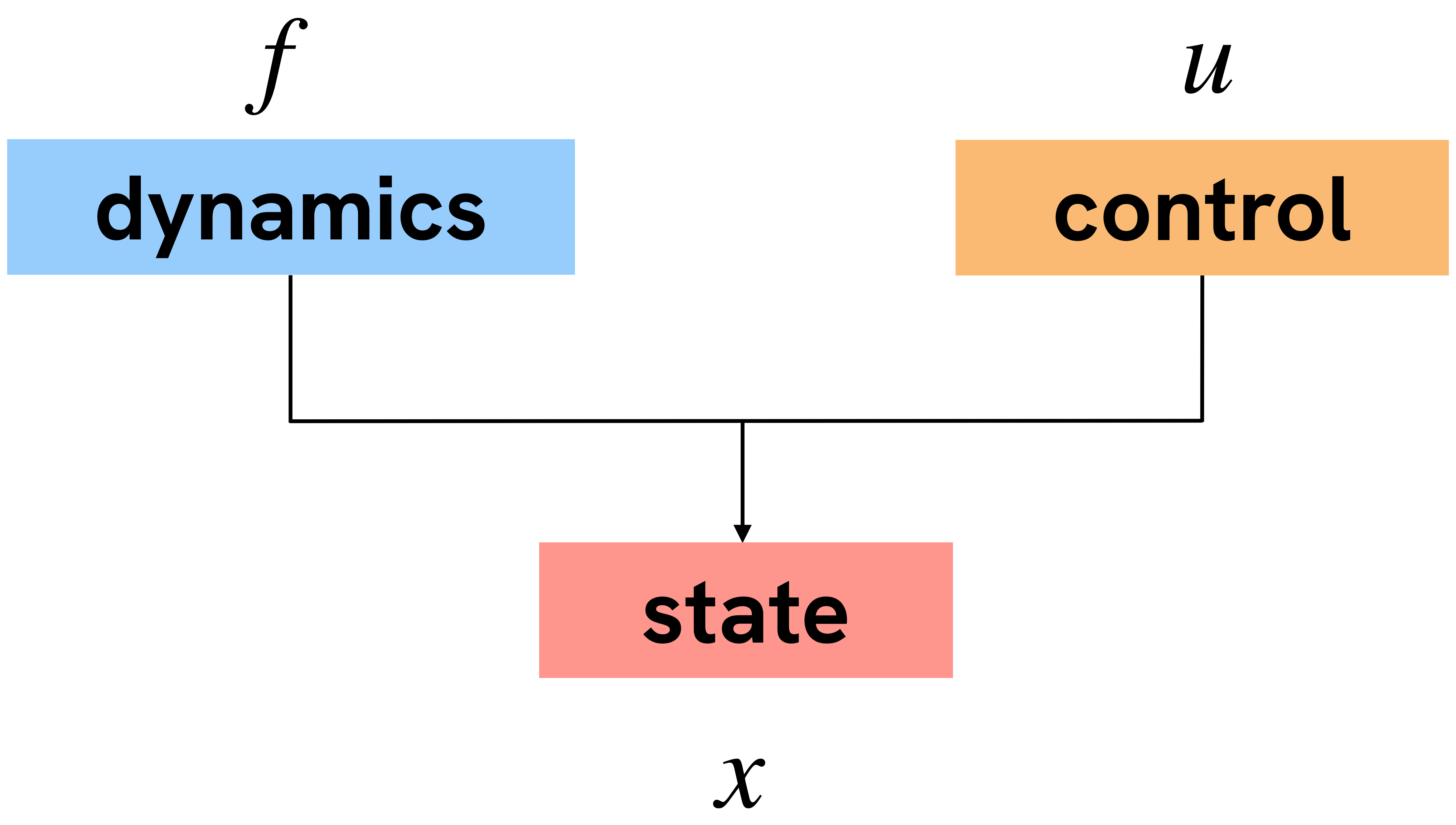
u control

→ state x

$$\dot{x} = \sum_{i=1}^{m} f_i(x) u_i$$

Control-affine system

Controllability



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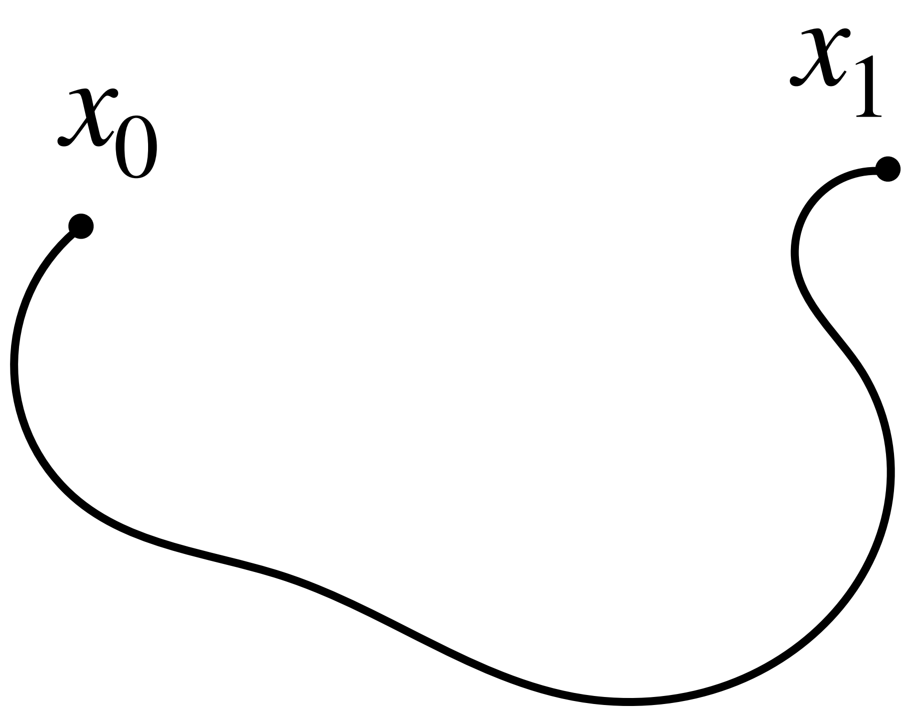
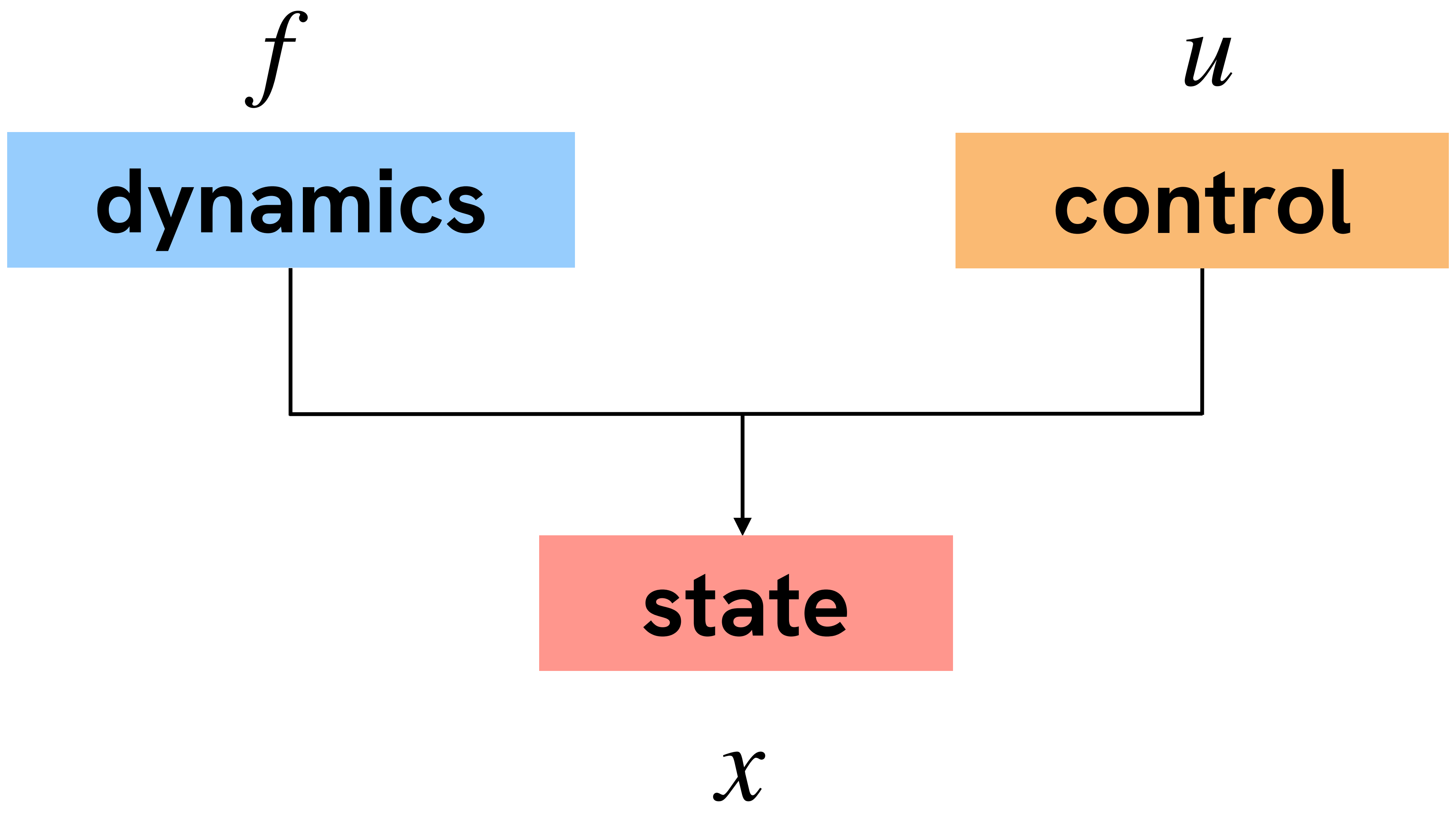
Control-affine system

Controllability: it is possible to go from x_0 to x_1 (for all x_0, x_1)



Controllability



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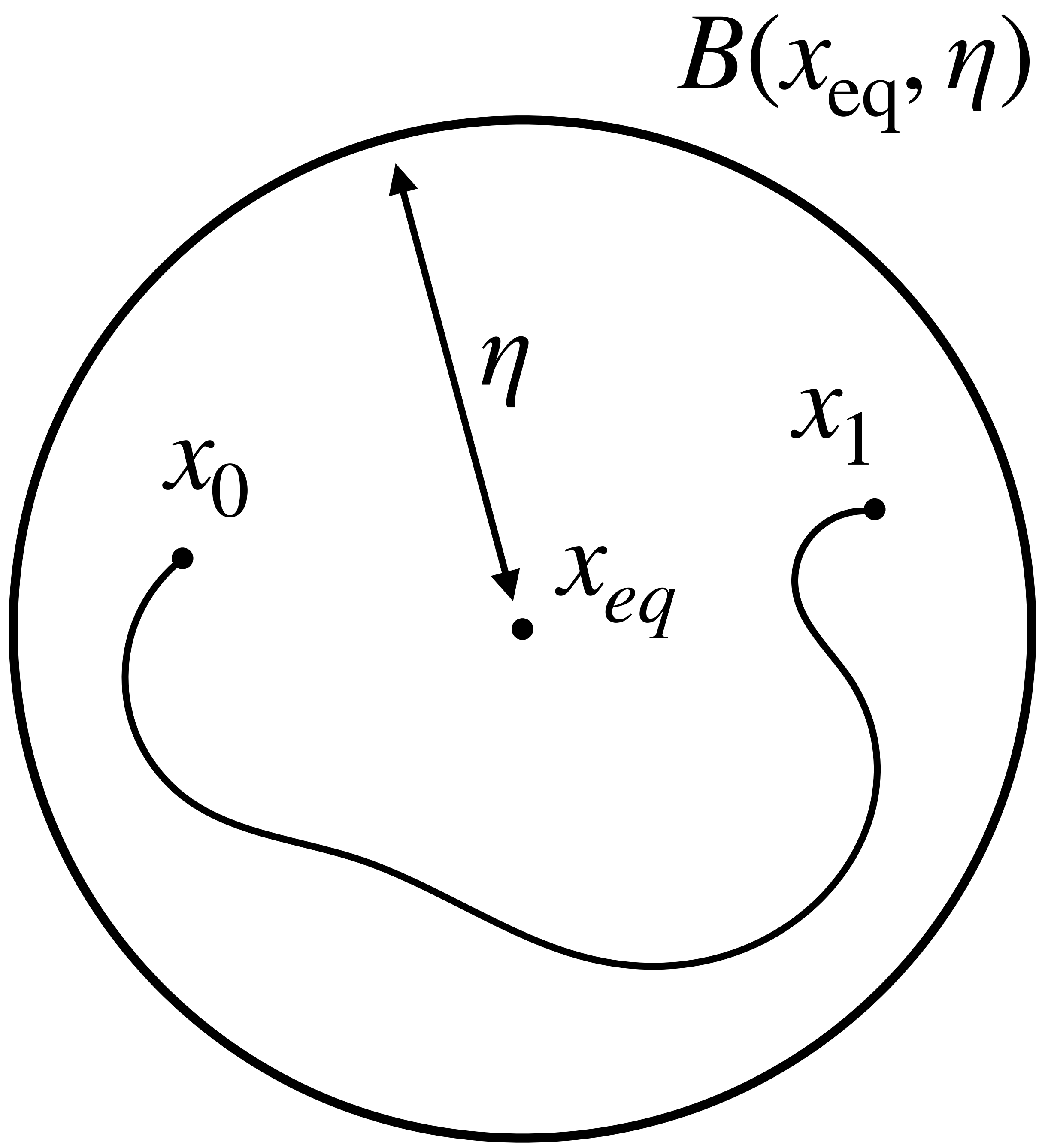
Control-affine system

Local controllability: it is possible to go from x_0 to x_1 **in a neighbourhood of an equilibrium**



STLC: in **small time** and with **small controls**

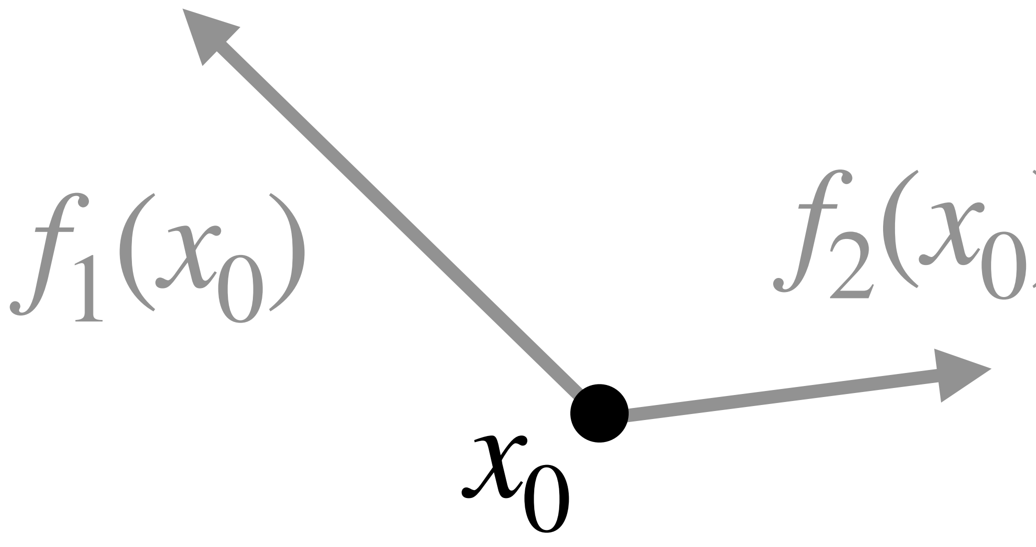
B-STLC: in **small time** and with **bounded controls**

Lie brackets: formal non-reciprocity

$$\dot{x} = \sum_{i=1}^{m} f_i(x) u_i$$

$m = 2$

$$\dot{x} = f_1(x)u_1 + f_2(x)u_2$$



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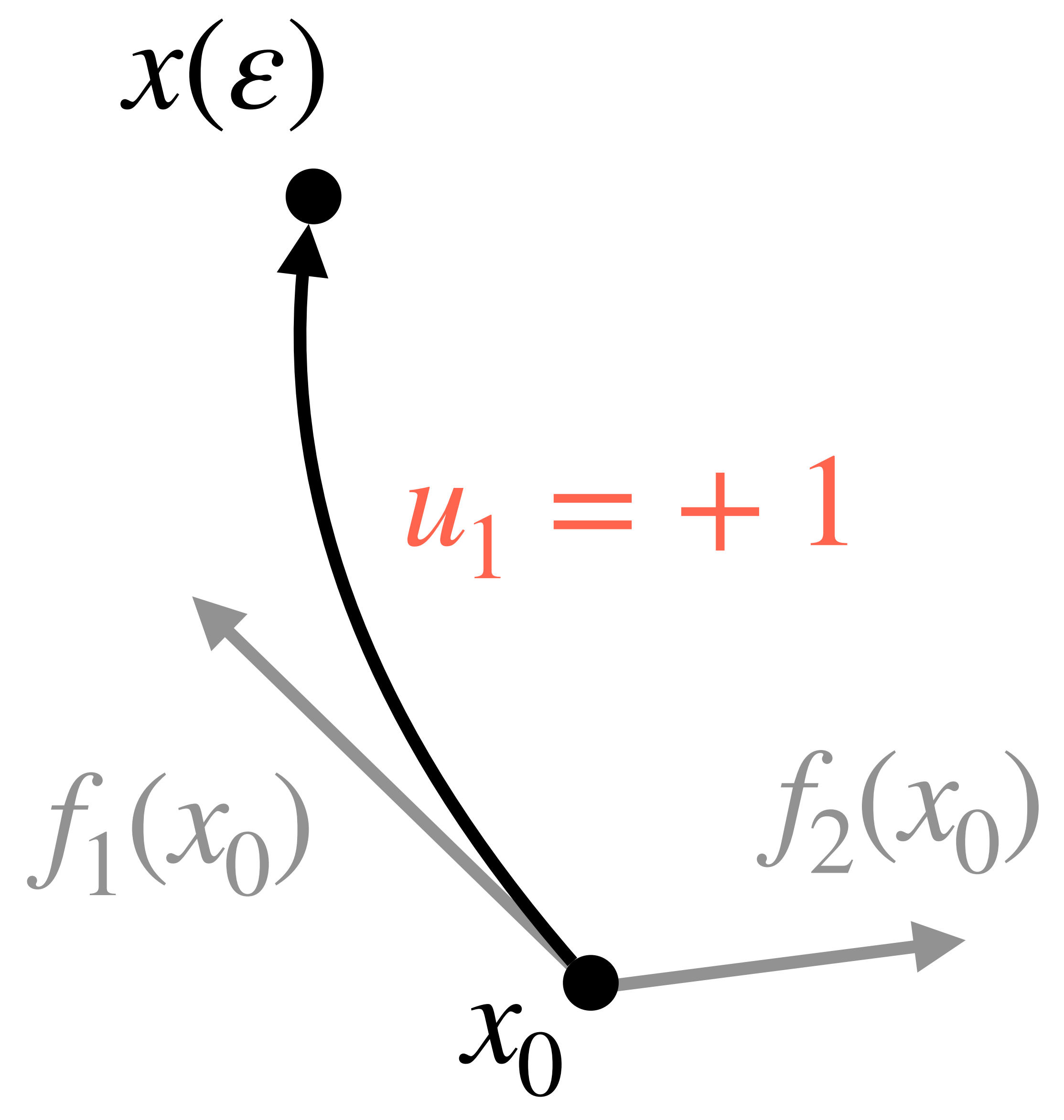

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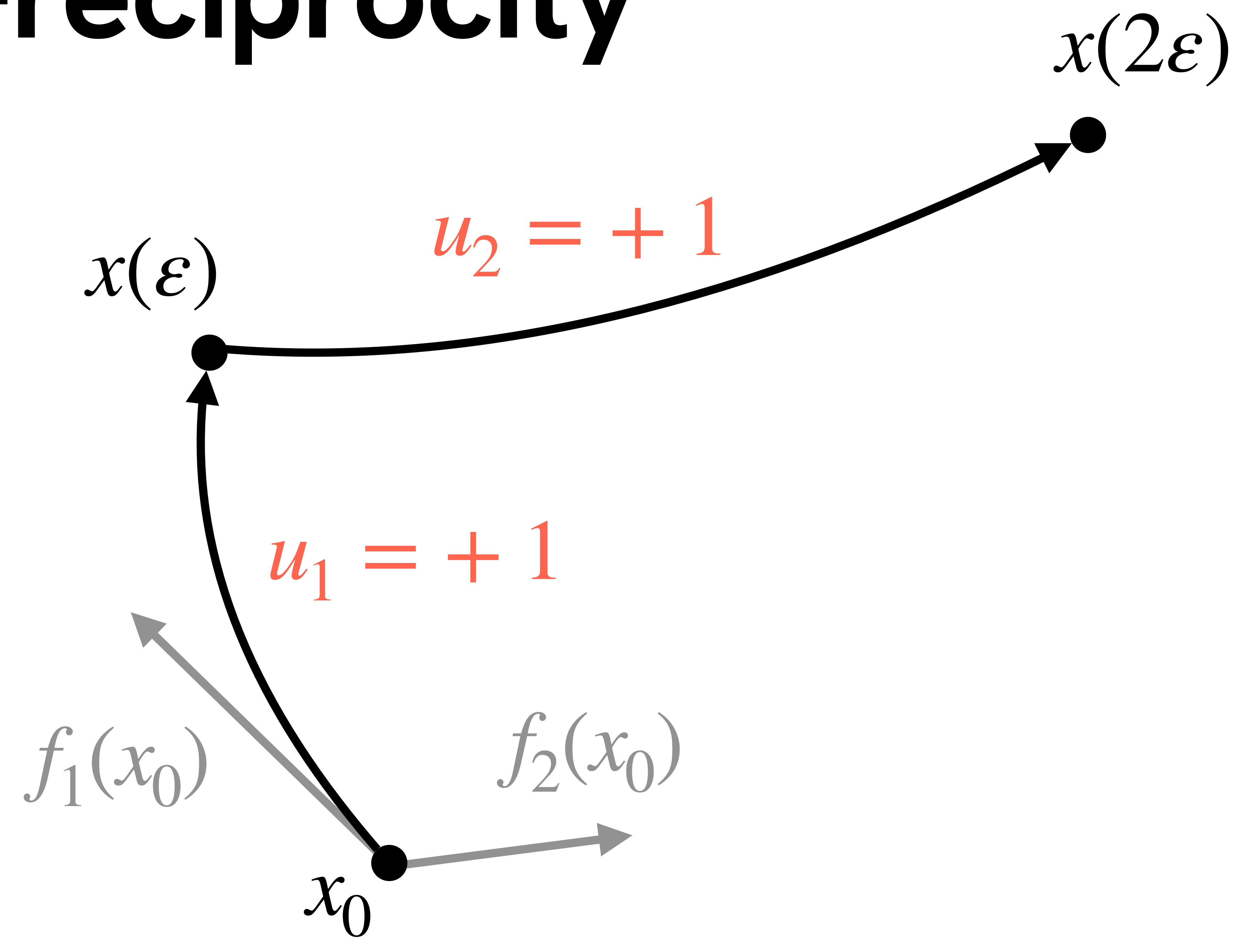

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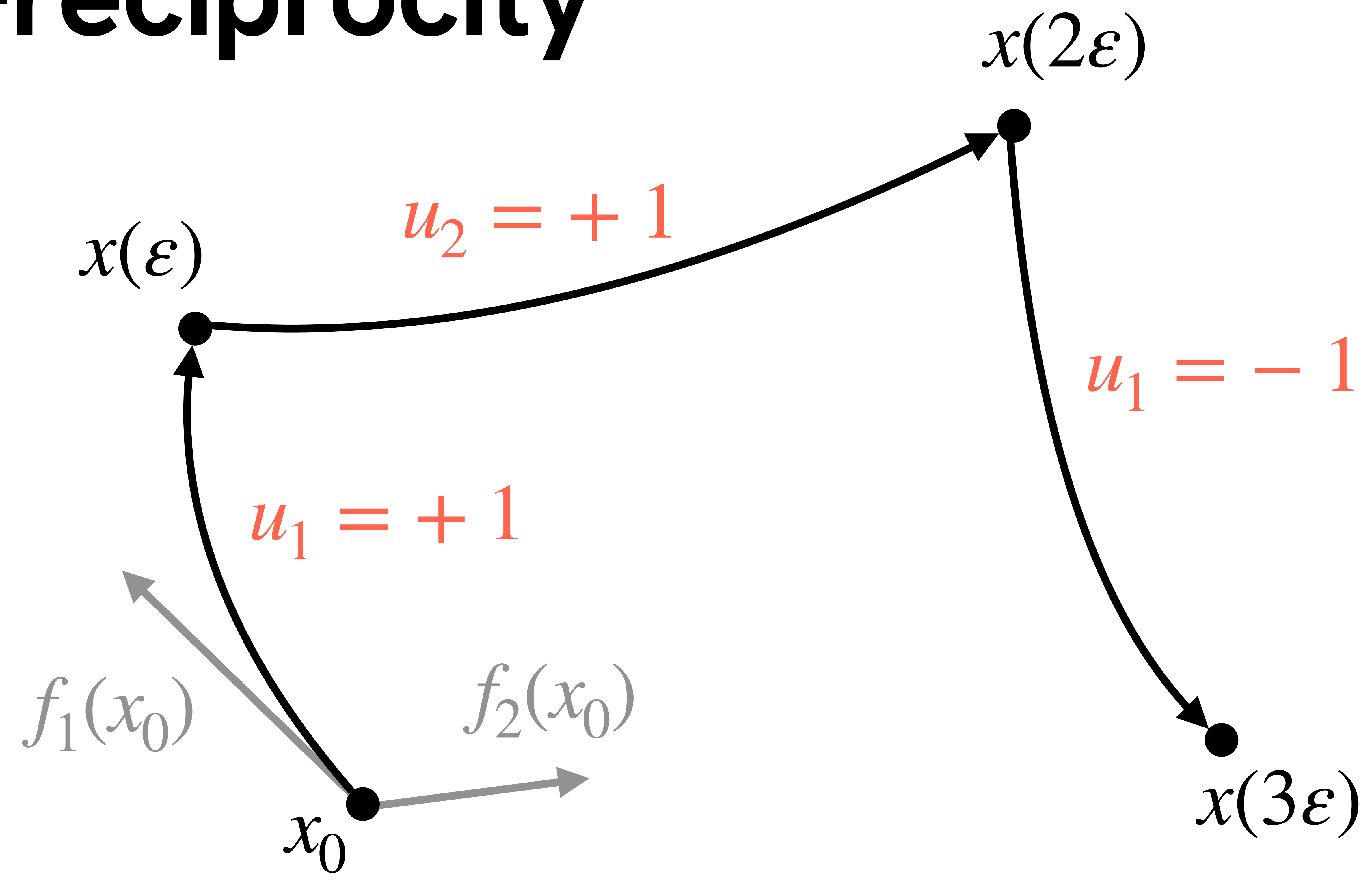

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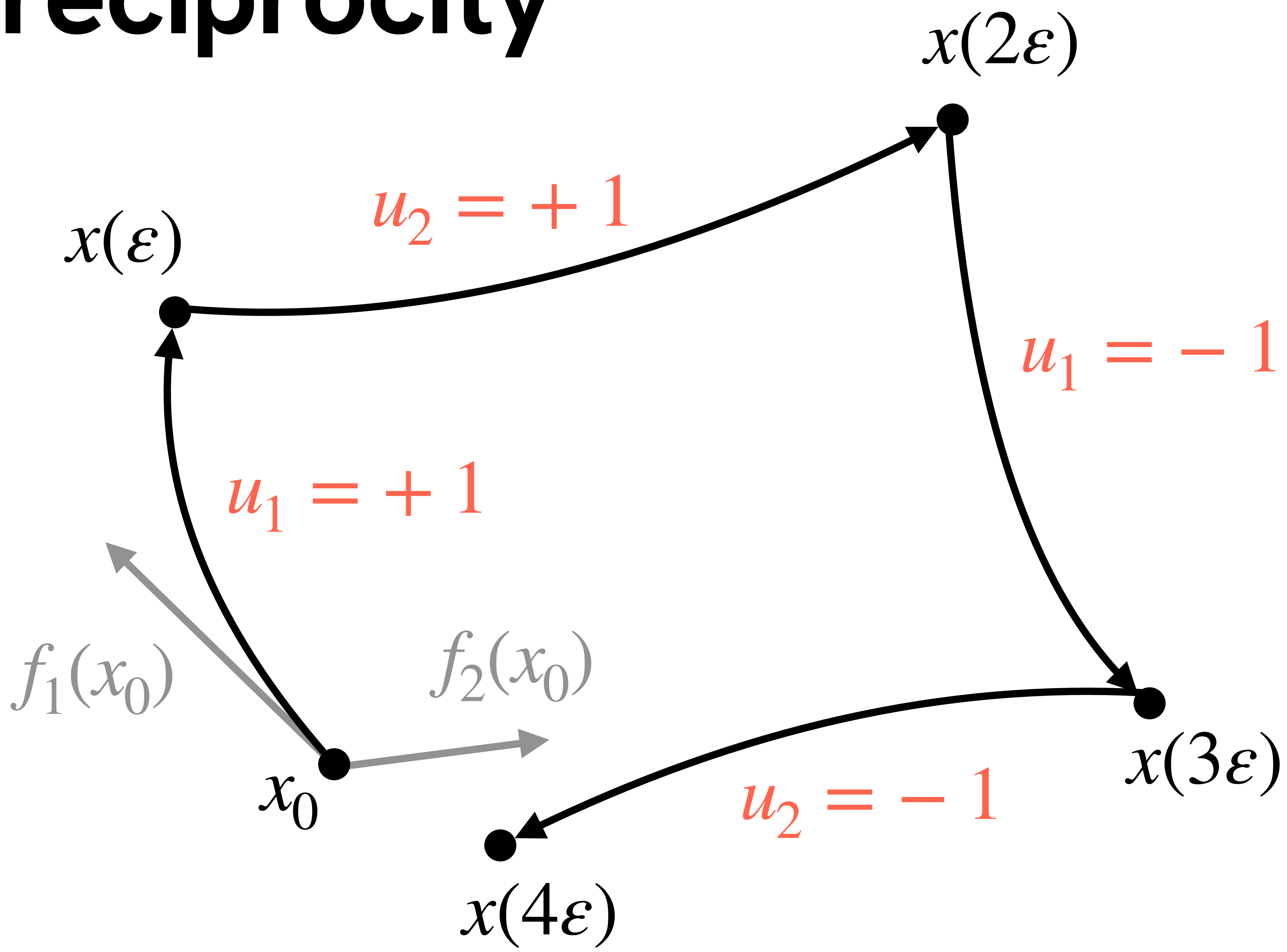

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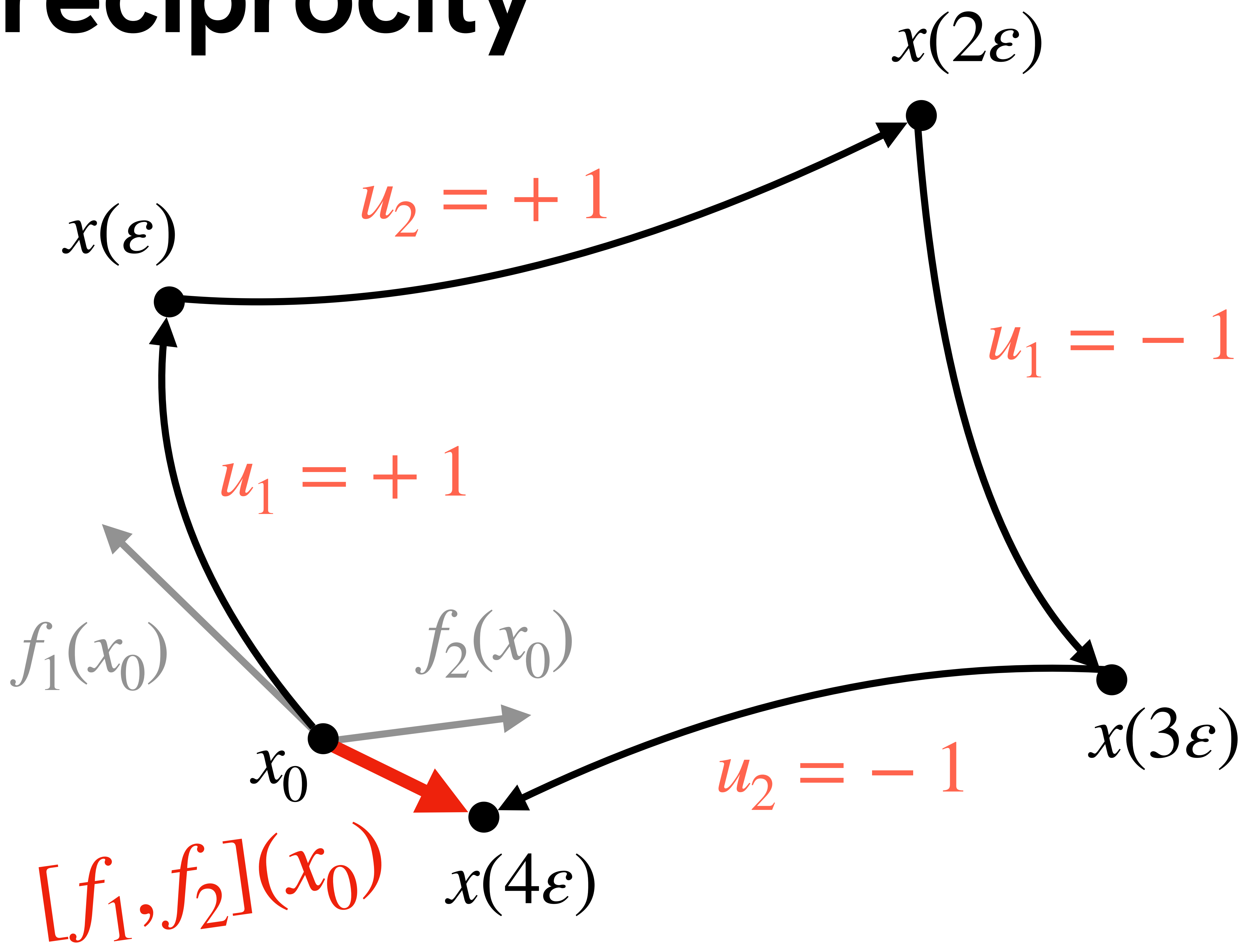

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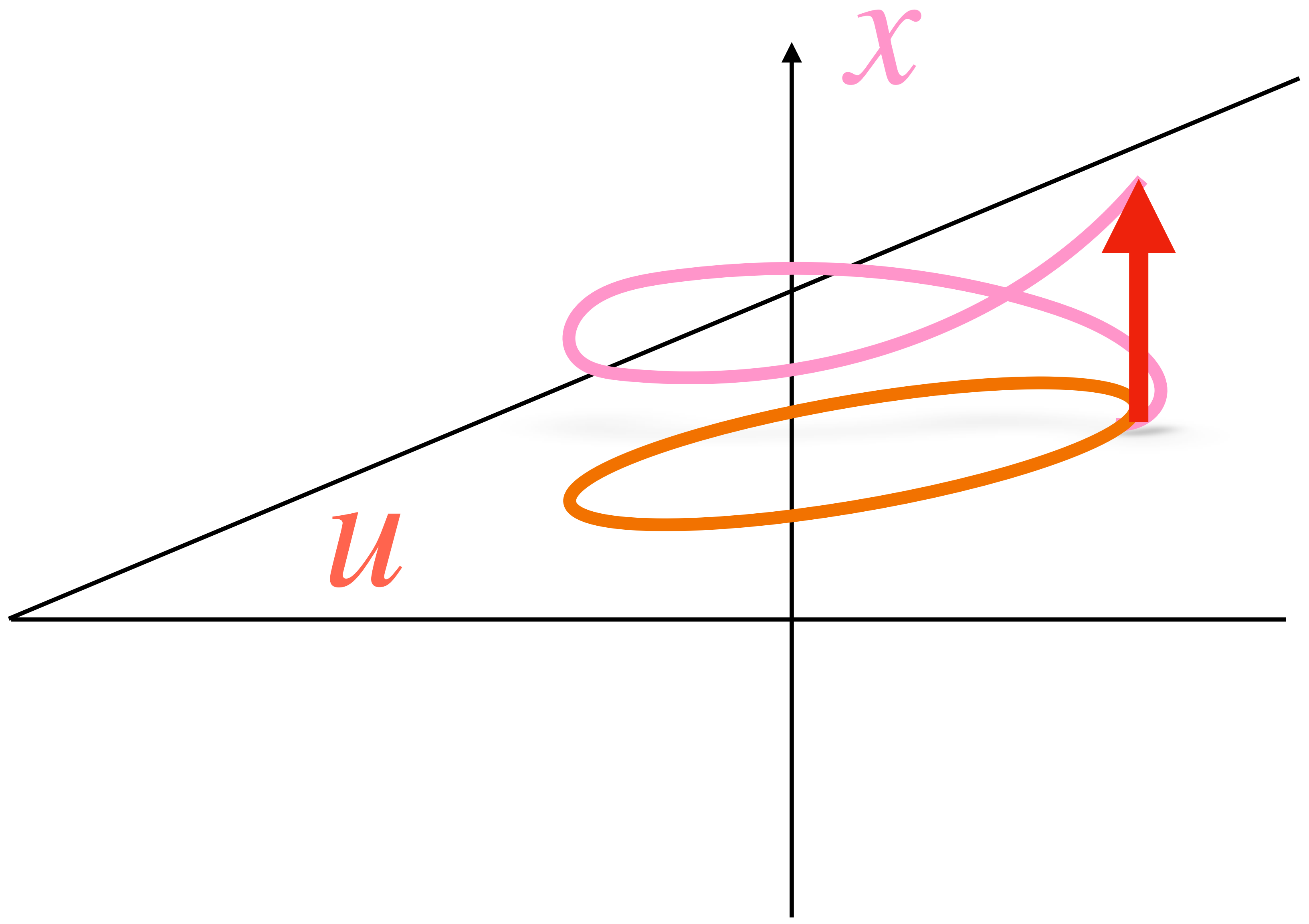

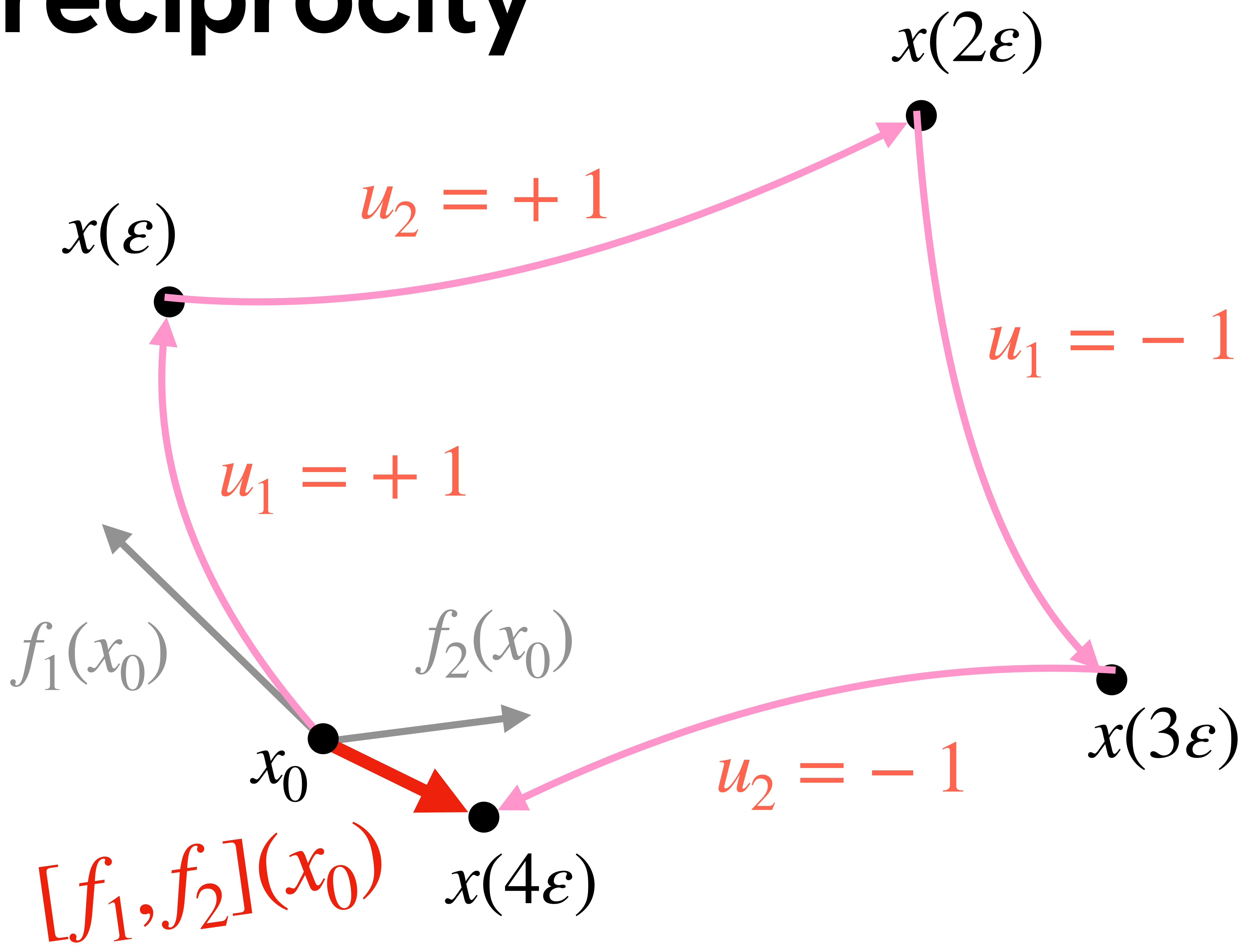

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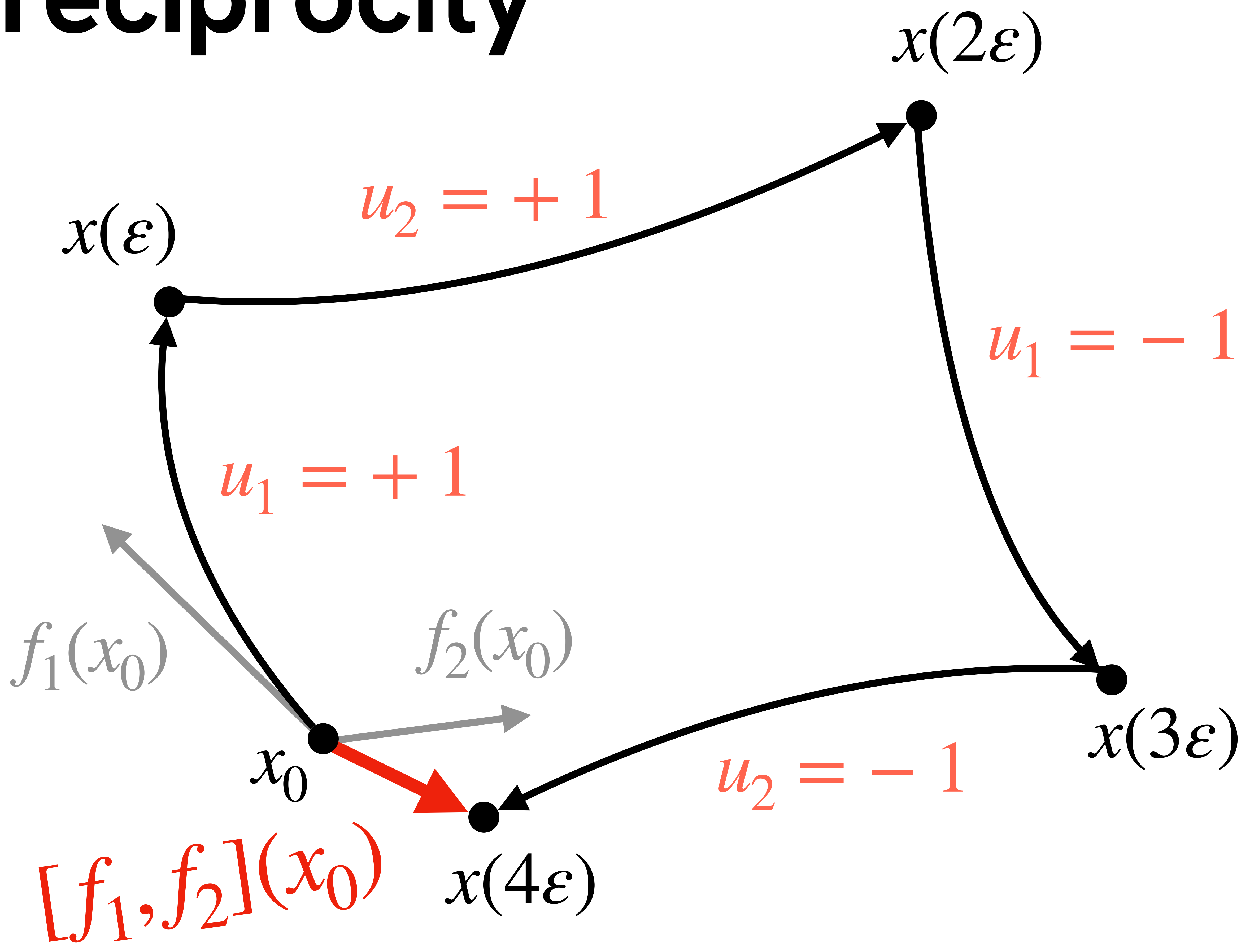
$$x(4\varepsilon) = x_0 + \varepsilon^2 [f_1, f_2](x_0) + o(\varepsilon^2)$$

$$[f_1, f_2](x) = \nabla f_2(x) f_1(x) - \nabla f_1(x) f_2(x)$$



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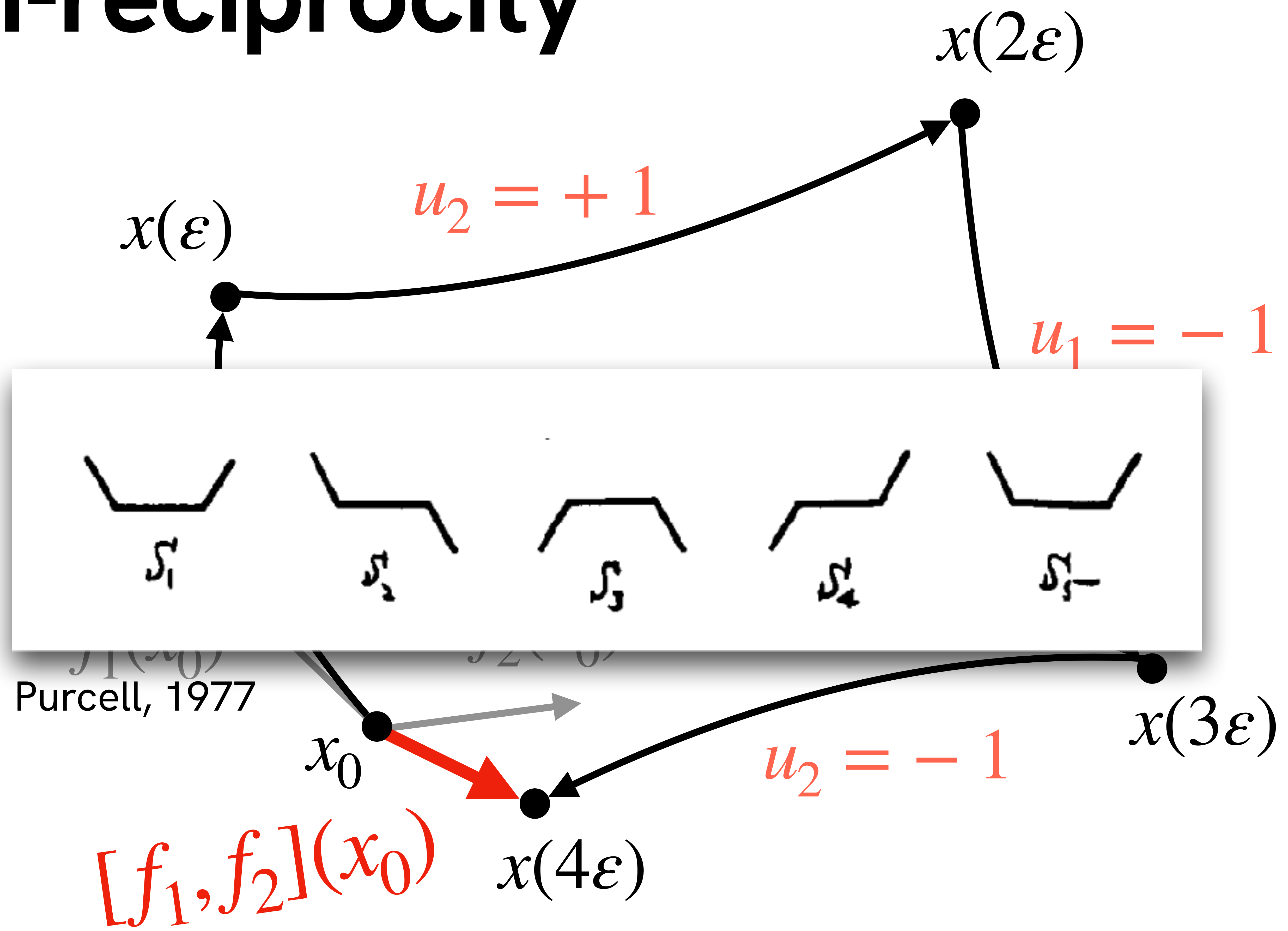

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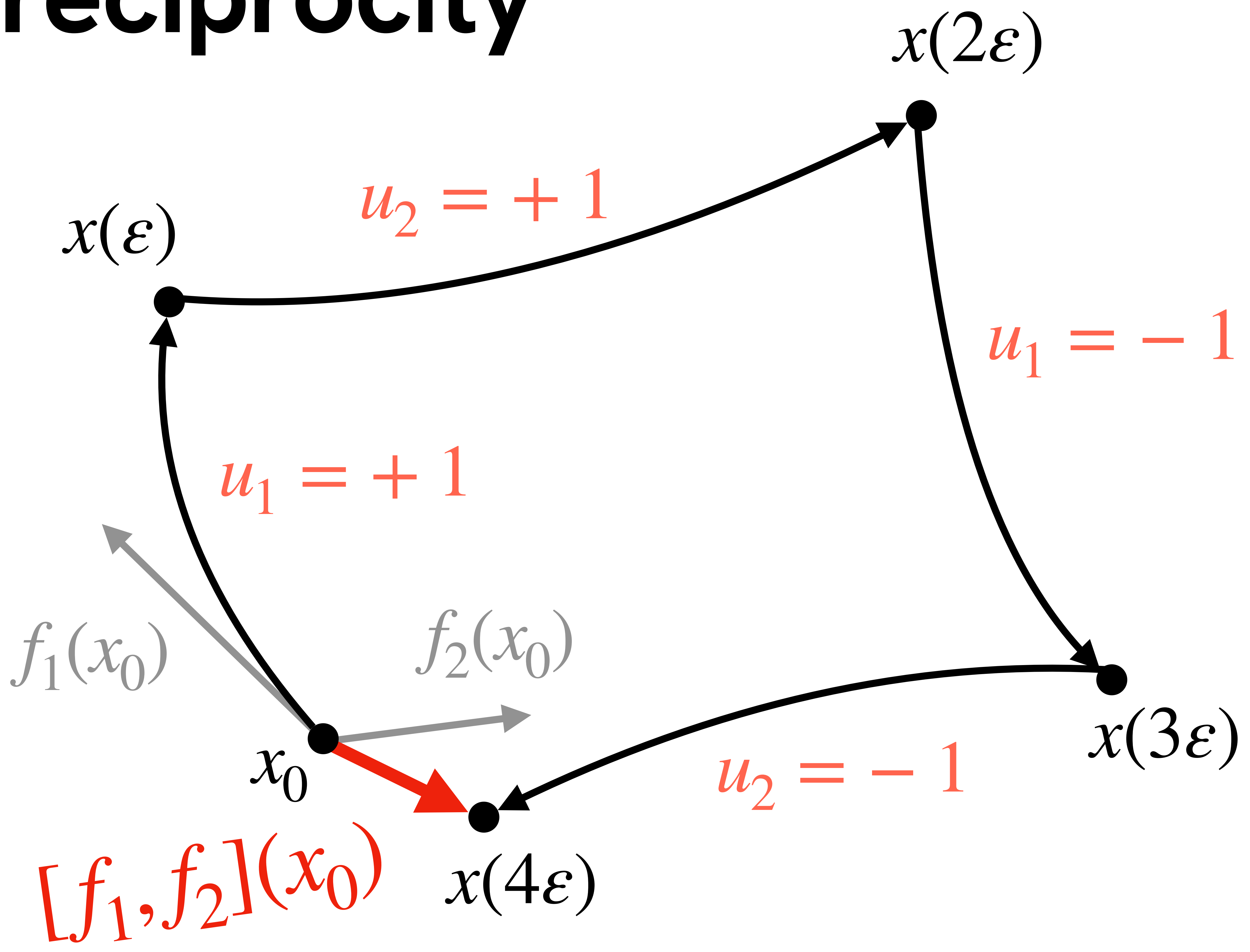

→ *We can generate new reachable directions given by the Lie brackets of the* f_i

→ of course, this can be iterated: $\big[[f_0, f_1], [f_0, [f_1, f_2]]\big]$, etc.

Conditions for controllability

$$\dot{x} = \sum_{i=1}^{m} f_i(x) u_i$$

<u>Definition</u>

Let $x_0 \in \mathbb{R}^n$. Let $\mathrm{Lie}(f_0, \ldots, f_m)$ be the set with all the iterated Lie brackets:

$$\mathrm{Lie}(f_0, \ldots, f_m) = \left\{ f_0, \ldots f_m, [f_0, f_1], \ldots, [f_0, f_m], \ldots, [[f_0, [f_0, f_i]], [f_j, f_k]], \ldots \right\}$$

Lie Algebra Rank Condition in the point x_0:

$$\mathrm{Span}\{g(x_0), g \in \mathrm{Lie}(f_1, \ldots, f_m)\} = \mathbb{R}^n.$$

→ *"The LARC is satisfied if Lie brackets generate all possible directions"*

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Theorem (Rashevski-Chow)

Lie Algebra Rank Condition at x_0 $\Leftrightarrow$ local controllability at x_0

LARC for all x_0 in $\mathbb{R}^n$ $\Leftrightarrow$ **controllable everywhere** in time T, for all T

Conditions for controllability

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Theorem (Rashevski-Chow)

Lie Algebra Rank Condition at x_0 ⇔ local controllability at x_0

LARC for all x_0 in $\mathbb{R}^n$ ⇔ **controllable everywhere** in time T, for all T

→ **Scallop theorem :** $m = 1$...

Controllability of microswimmers: an overview

hydrodynamics

rigid motion

deformation

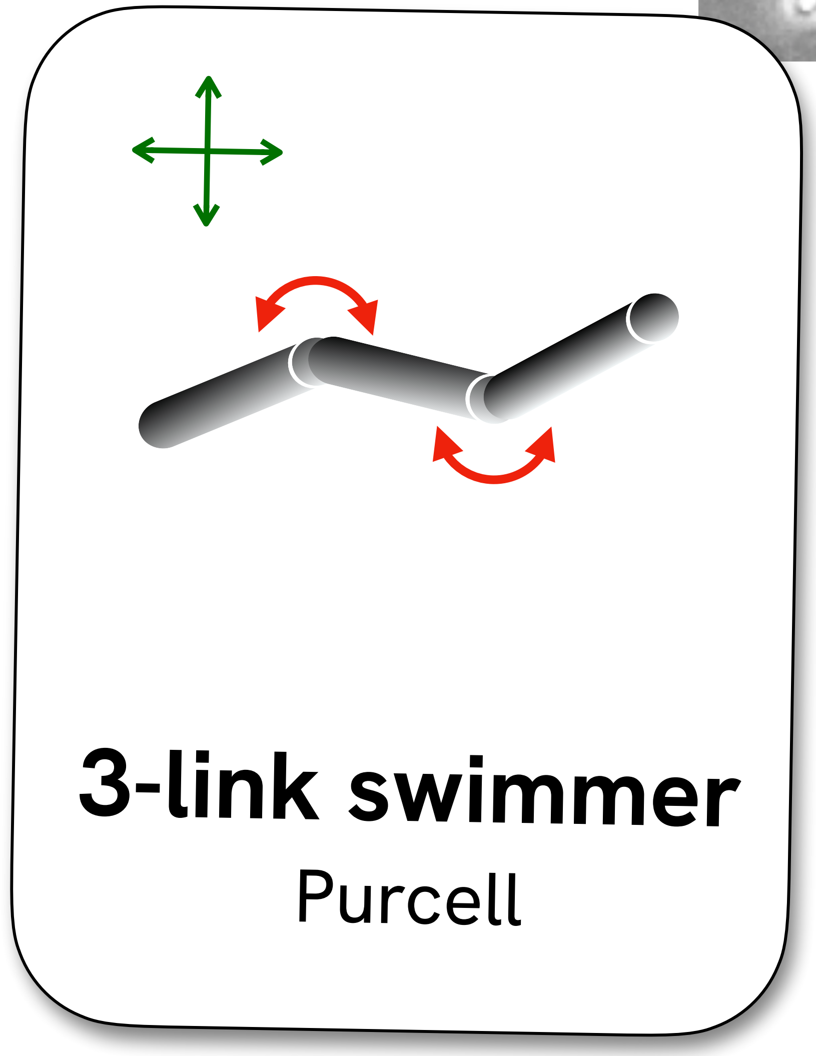
$$\dot{\mathscr{X}} = \mathbf{N}(\mathscr{S})\dot{\mathscr{S}}$$

The swimming equation is a **driftless control-affine system**

→ **Rashveski-Chow theorem**

→ **many results of controllability**

flagellar motion



Giraldi et al., CDC, 2013

Marchello et al. Acta Appl. Math., 2022

Controllability of microswimmers: an overview

hydrodynamics

rigid motion

deformation

$$\dot{\mathscr{X}} = \mathbf{N}(\mathscr{S})\dot{\mathscr{S}}$$

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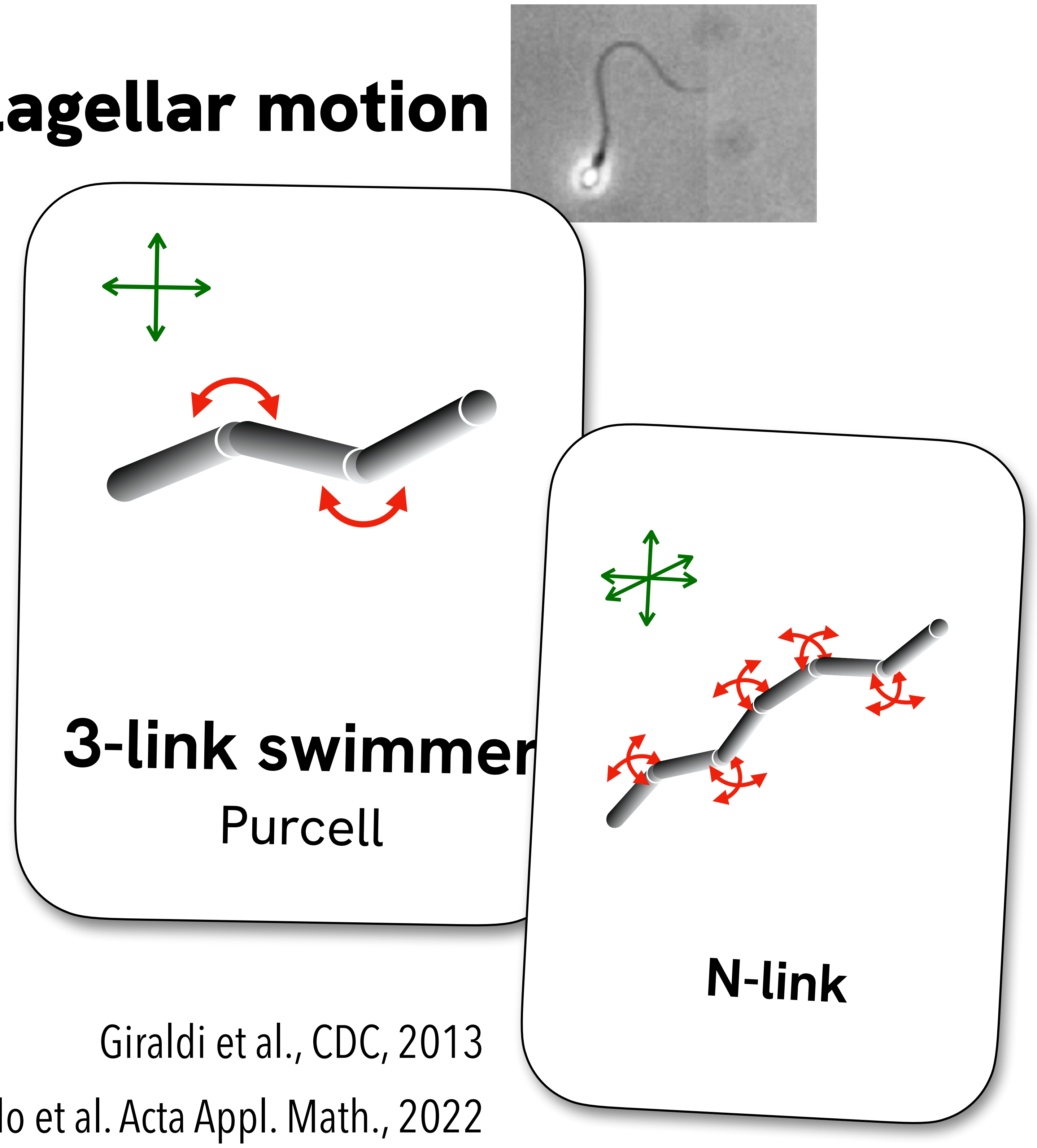

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rigid motion, hydrodynamics, deformation

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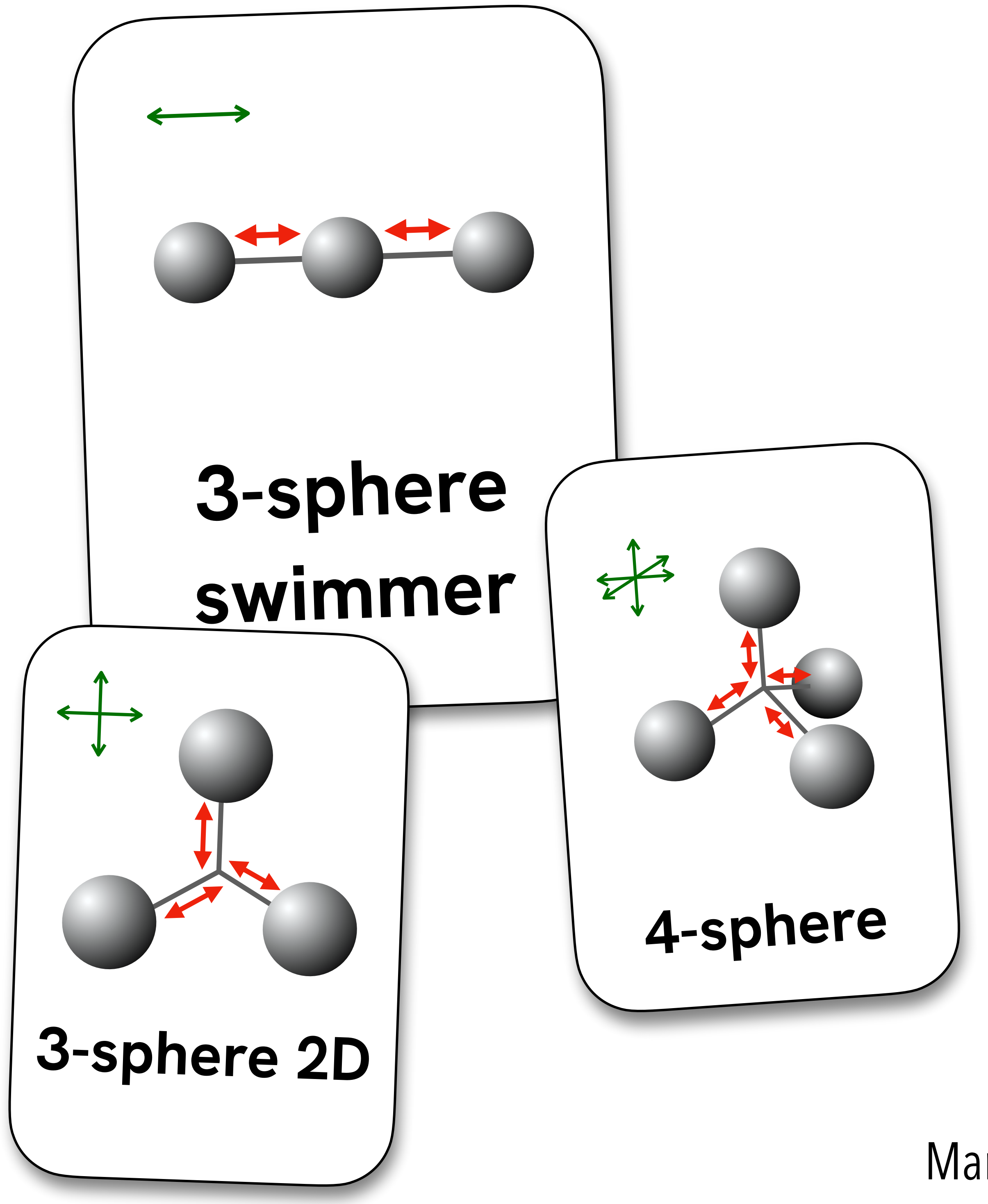
minimal swimmers

Alouges et al., Discrete Contin. Dyn. Syst. 2013



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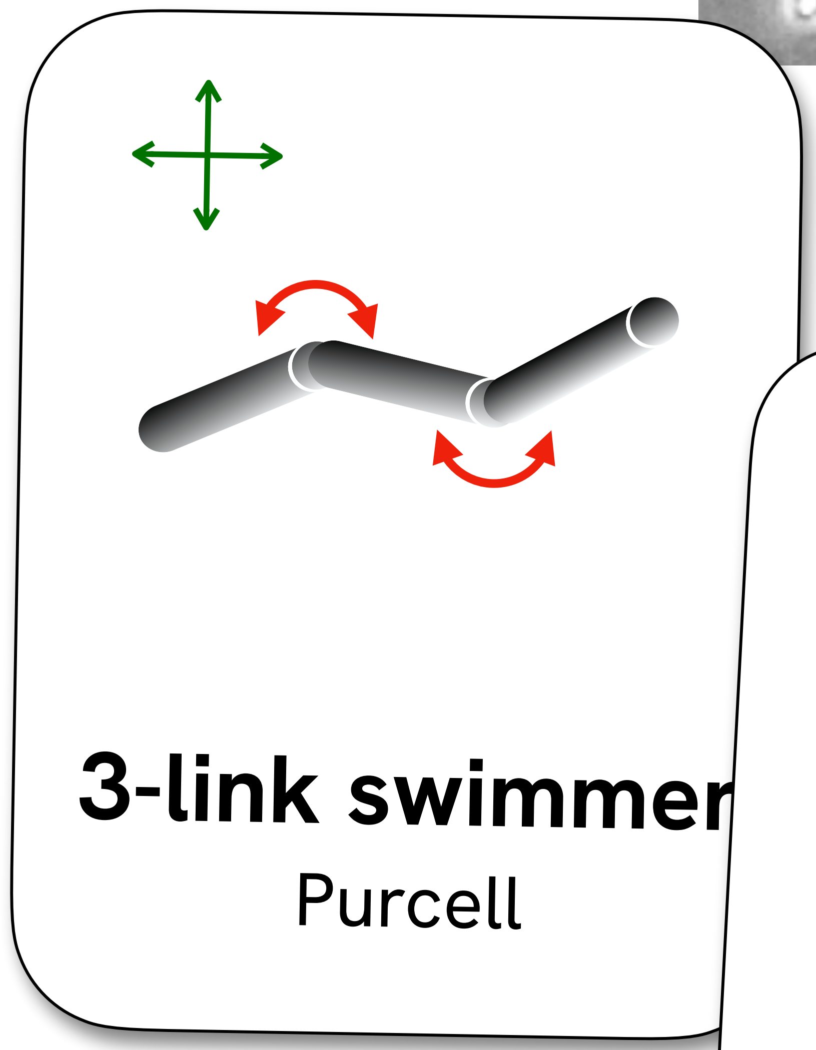

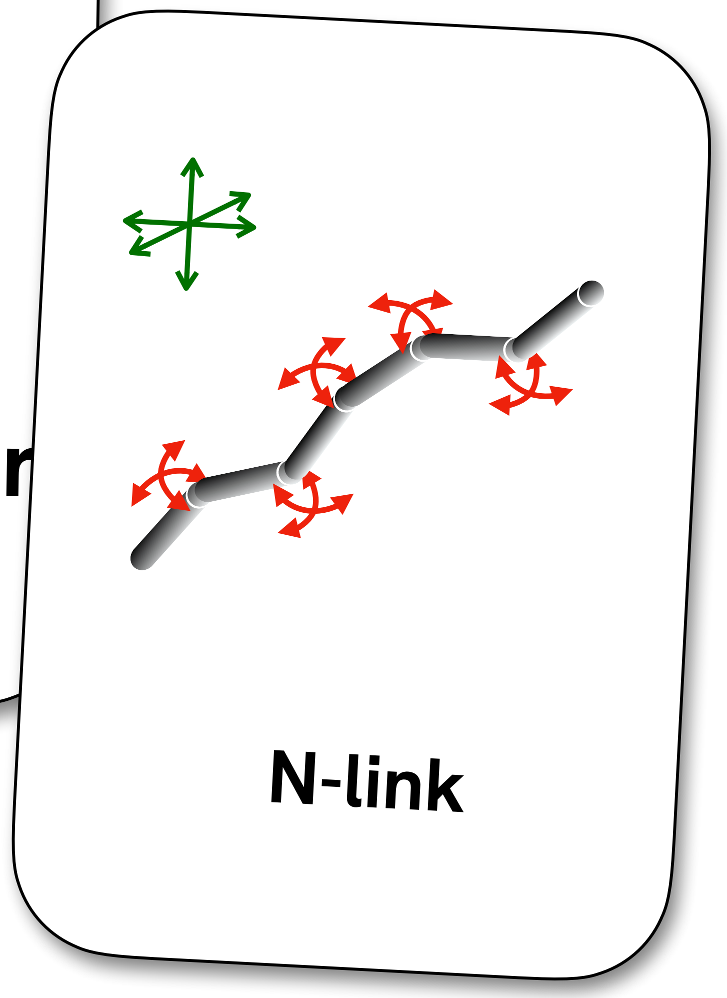

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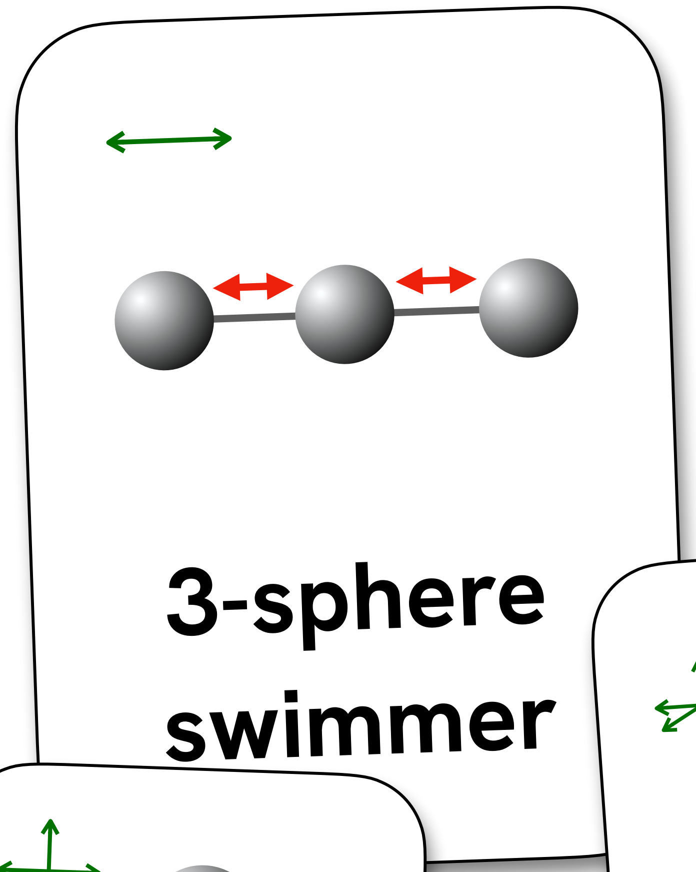

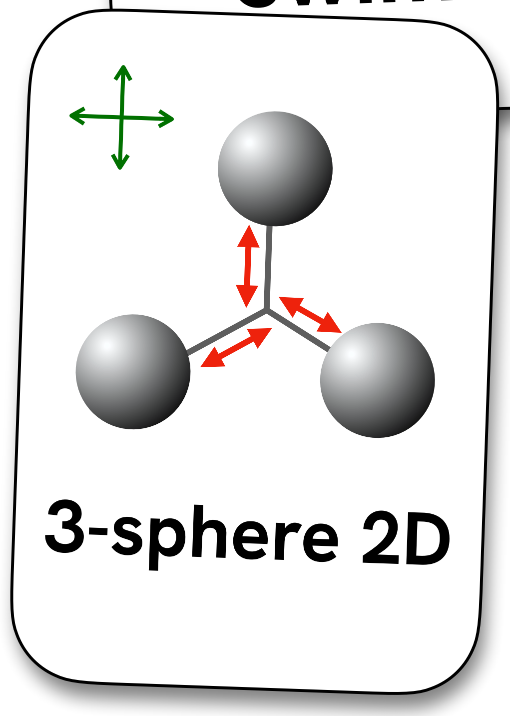

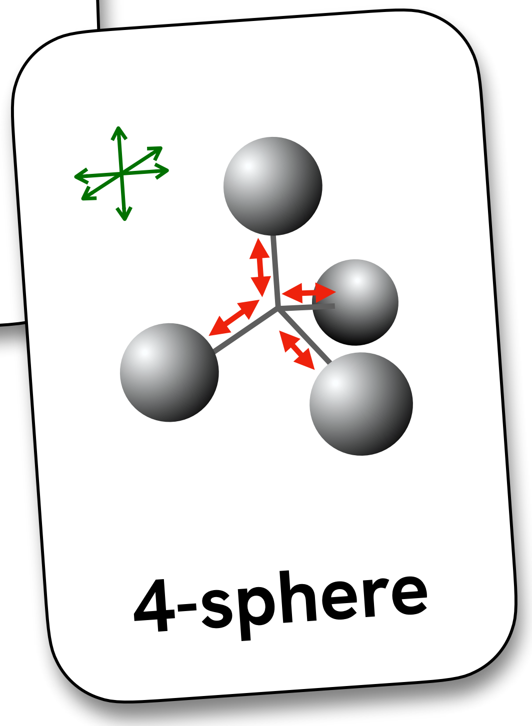

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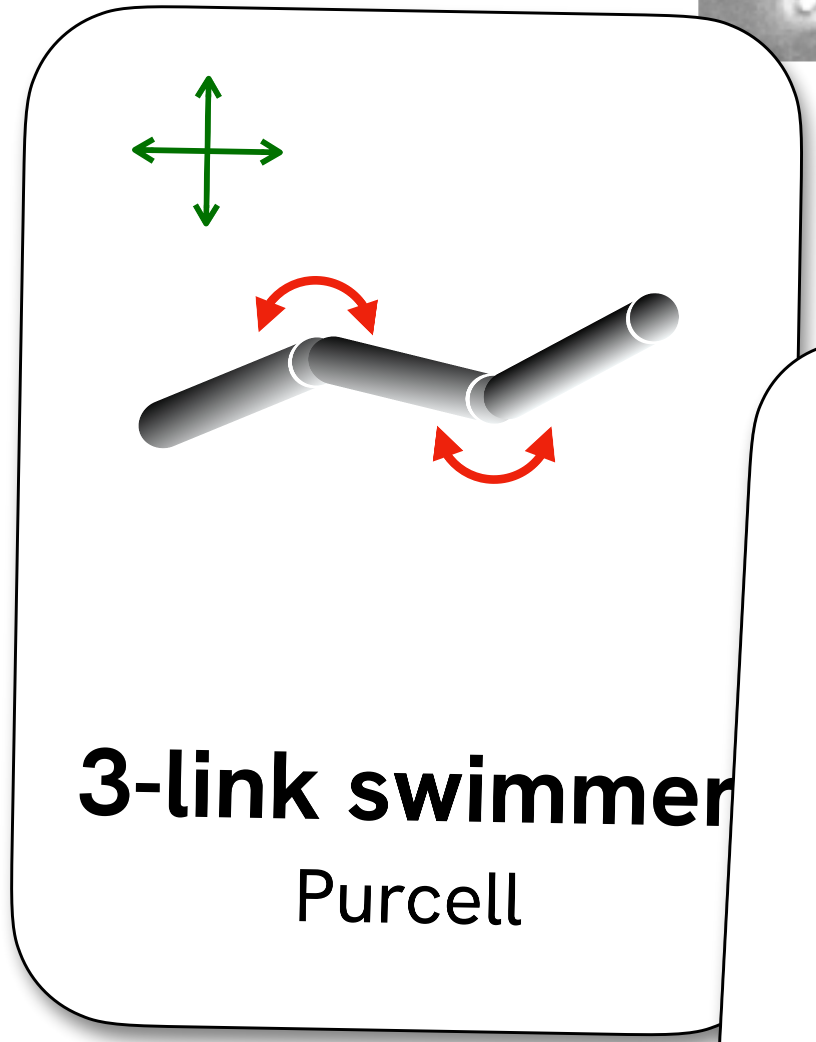

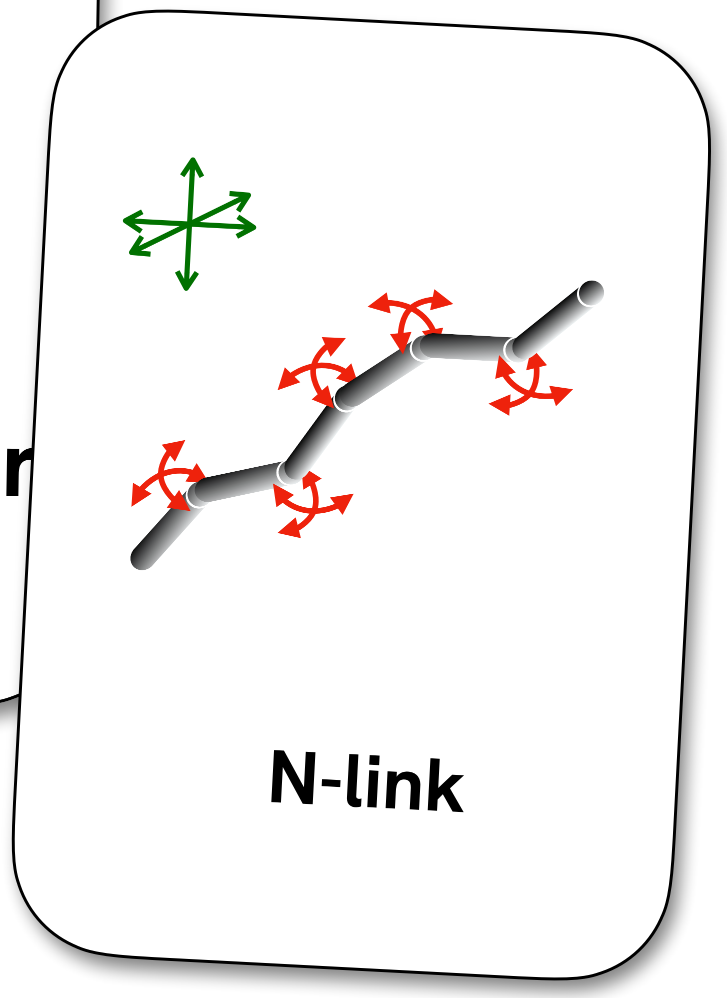

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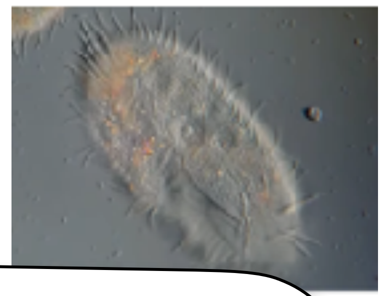
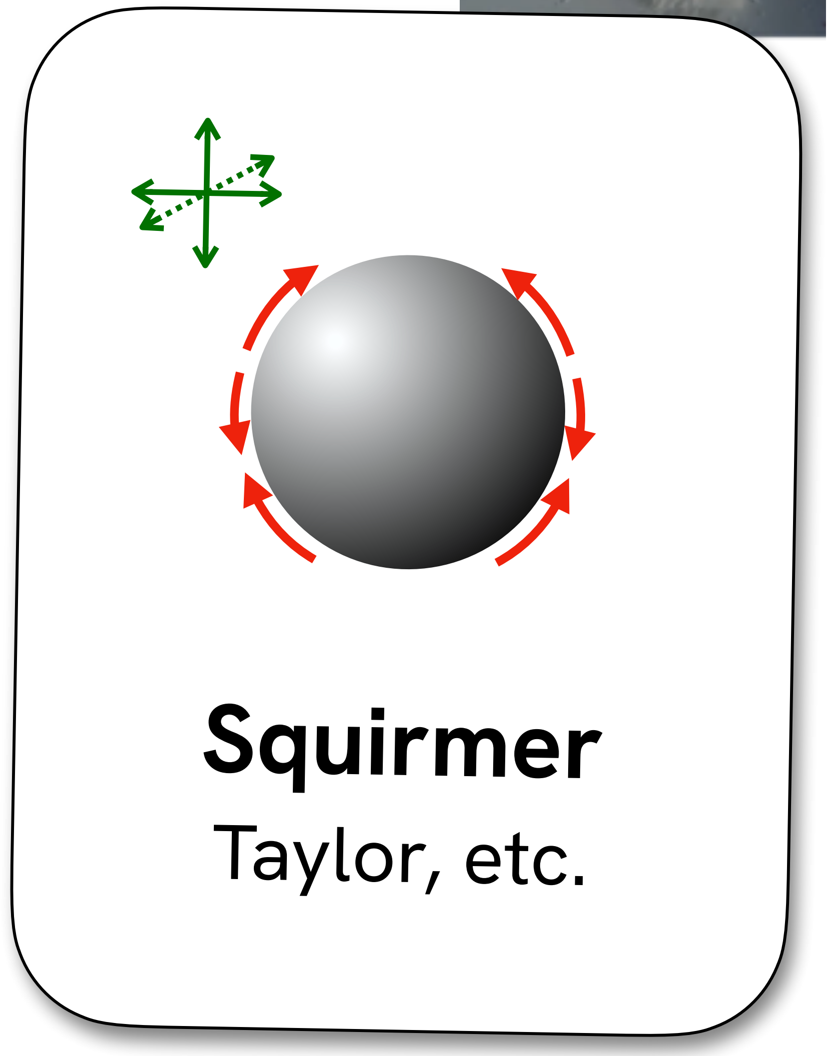
Marchello et al. Acta Appl. Math., 2022

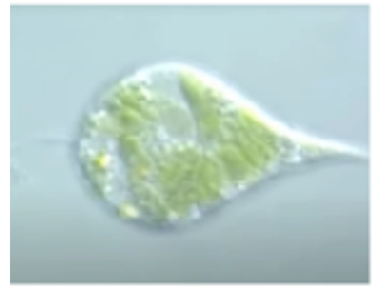
ciliate motion



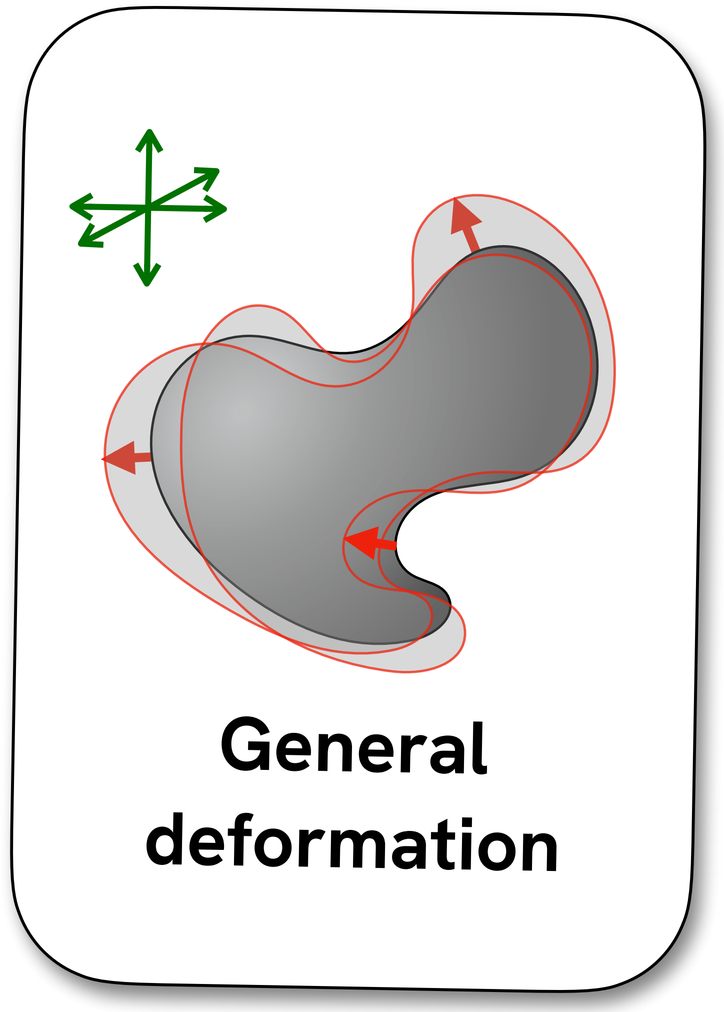


Lohéac, Takahashi, ESAIM:COCV, 2020

euglenoid motion



Lohéac, Munnier, ESAIM:COCV, 2014



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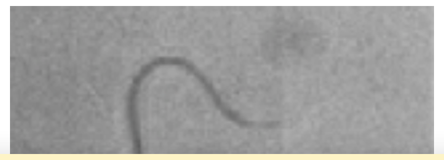

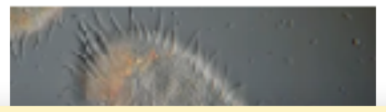

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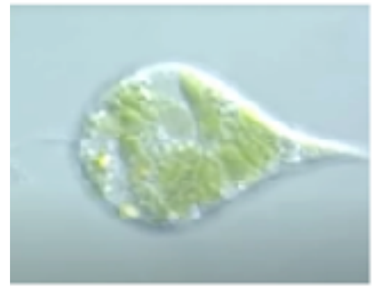
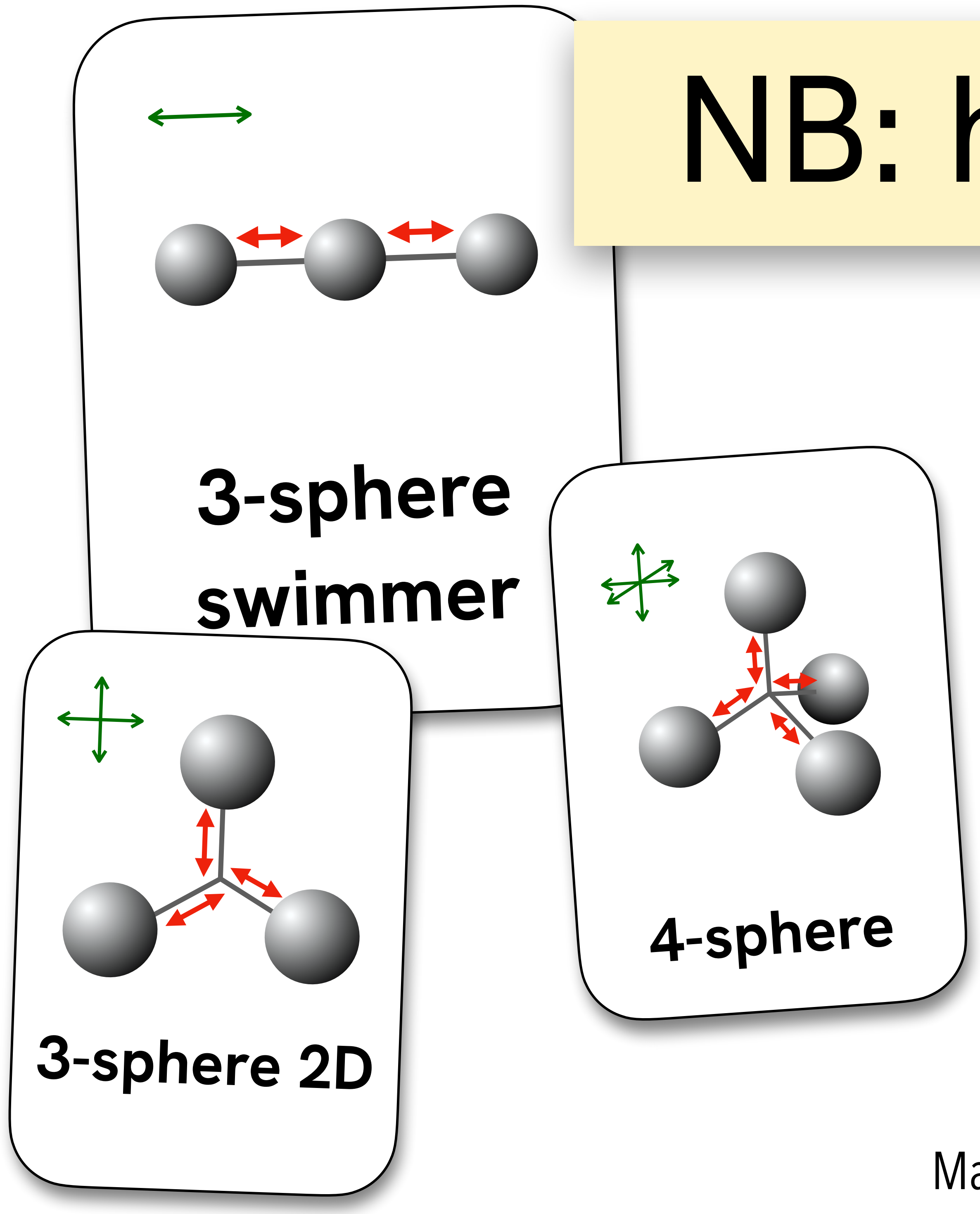
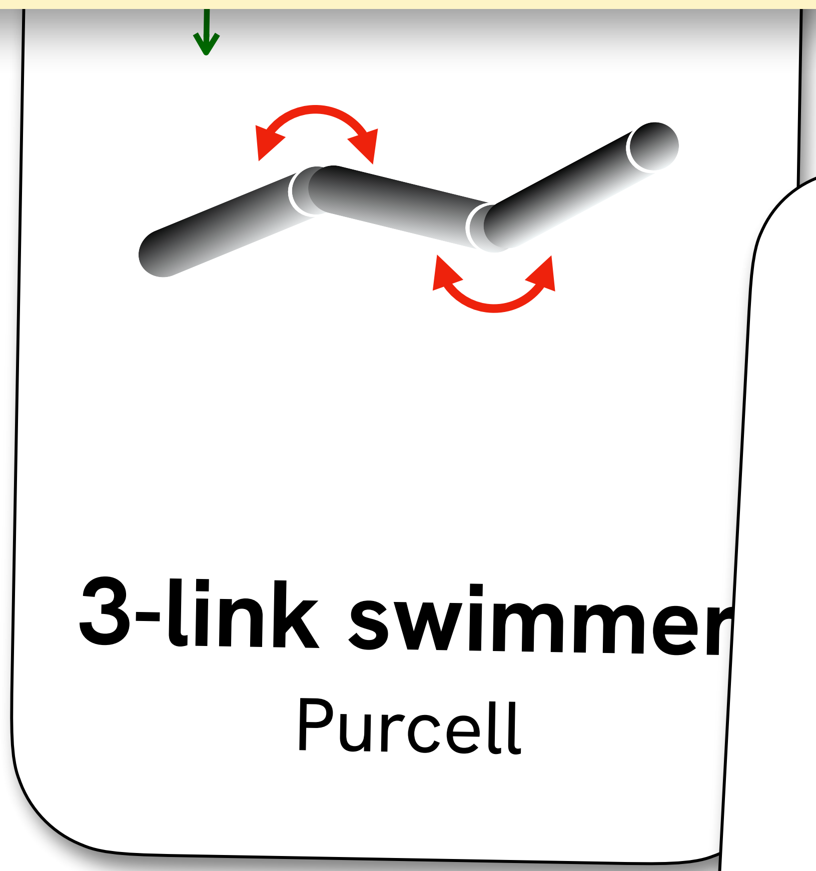
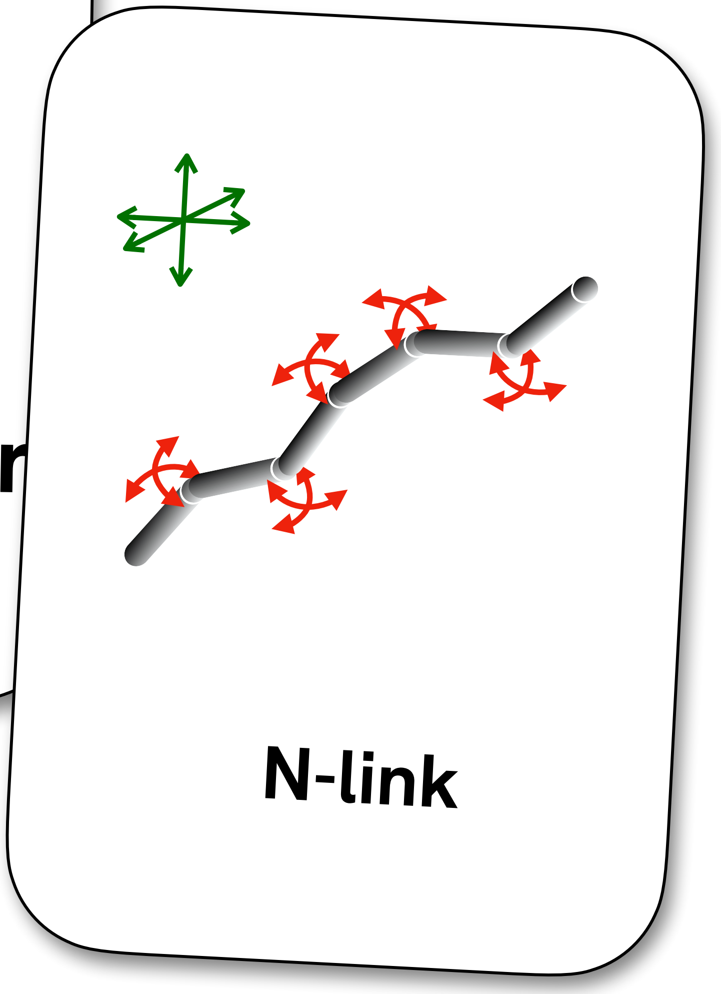


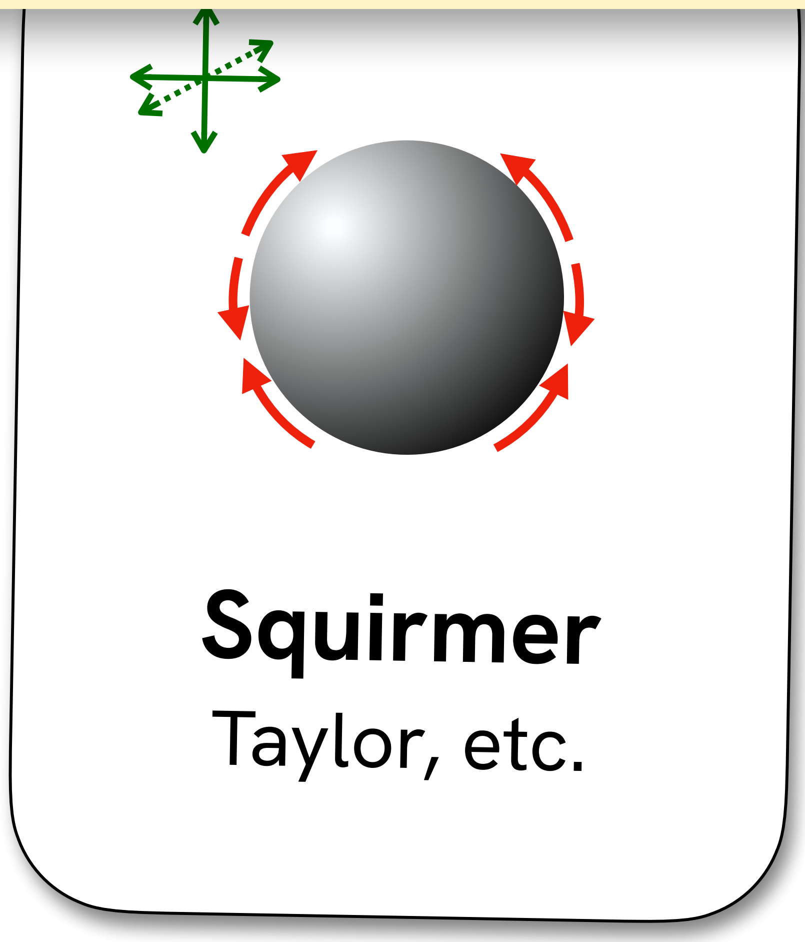
NB: here, more d.o.f. makes it « easier »!

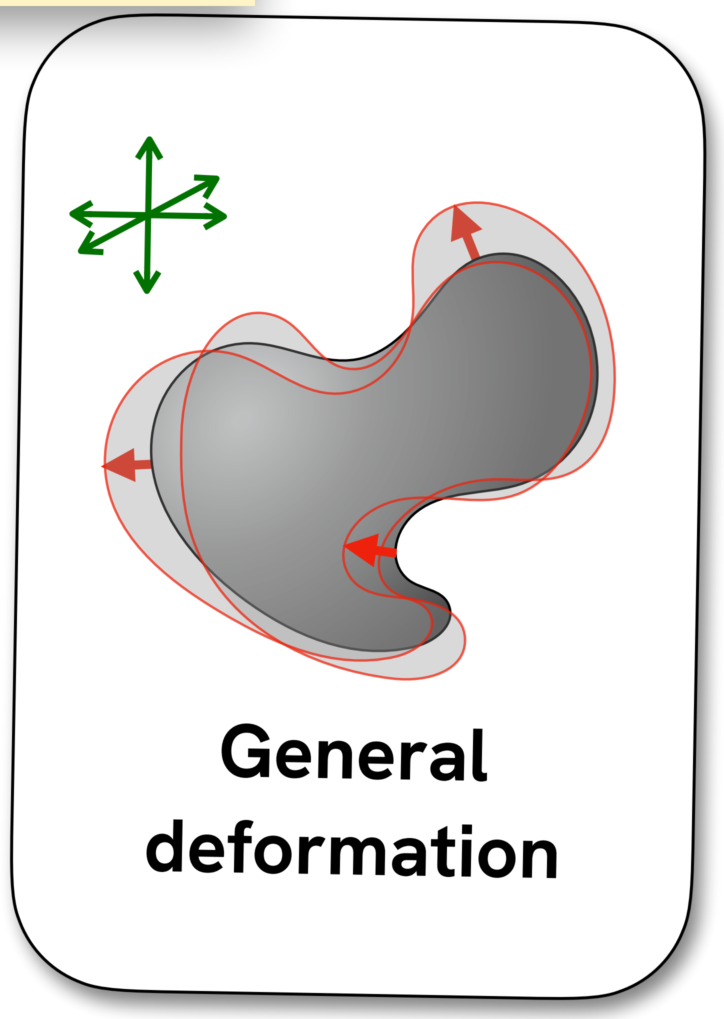
ier, ESAIM:COCV, 2014











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Beyond the canonical swimming model

Questionable assumptions

1. Newtonian flow

2. Free-space swimming

3. Quiescent fluid

4. No external effects

5. Control by deformation

Beyond the canonical swimming model

$$\cancel{\dot{\mathscr{X}} = \mathbf{N}(\mathscr{S})\dot{\mathscr{S}}}$$

1. Non-Newtonian flow

$$\longrightarrow \begin{pmatrix} \dot{\mathscr{X}} \\ \dot{\mathscr{S}} \end{pmatrix} = \begin{pmatrix} \mathrm{N}(\mathscr{S}, \boxed{t}) \\ \mathrm{Id} \end{pmatrix} \dot{\mathscr{S}}$$

Memory effect

or non-linear, more realistically

- Virtually nothing on controllability

2. Free-space swimming

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Memory effect

2. Obstacles

$$\longrightarrow \begin{pmatrix} \dot{\mathscr{X}} \\ \dot{\mathscr{S}} \end{pmatrix} = \begin{pmatrix} \mathbf{N}(\mathscr{S}, \mathscr{X}) \\ \mathrm{Id} \end{pmatrix} \dot{\mathscr{S}}$$

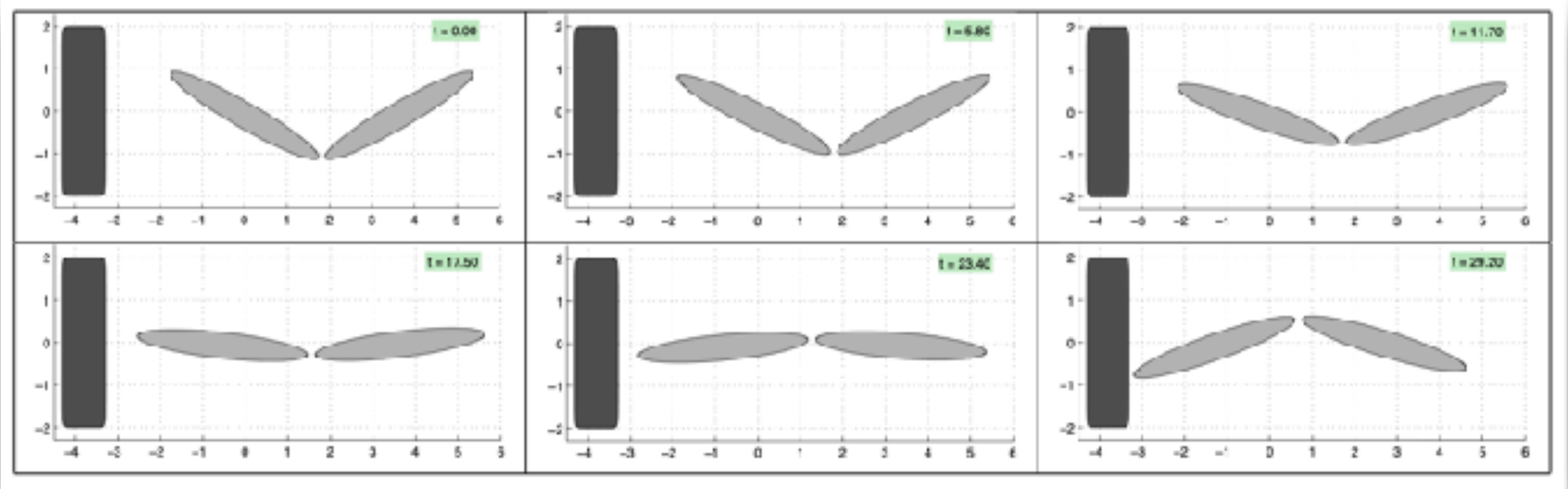
Dependence in space

3. Quiescent fluid

4. No external effects

5. Control by deformation

- **Near a wall**: Chambrion & Munnier 2010; Alouges & Giraldi 2013
- **Multiple swimmers**: Walker, M. et al. 2022, Zoppelllo, Shum et al. 2024



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Dependence in space

3. Background flow

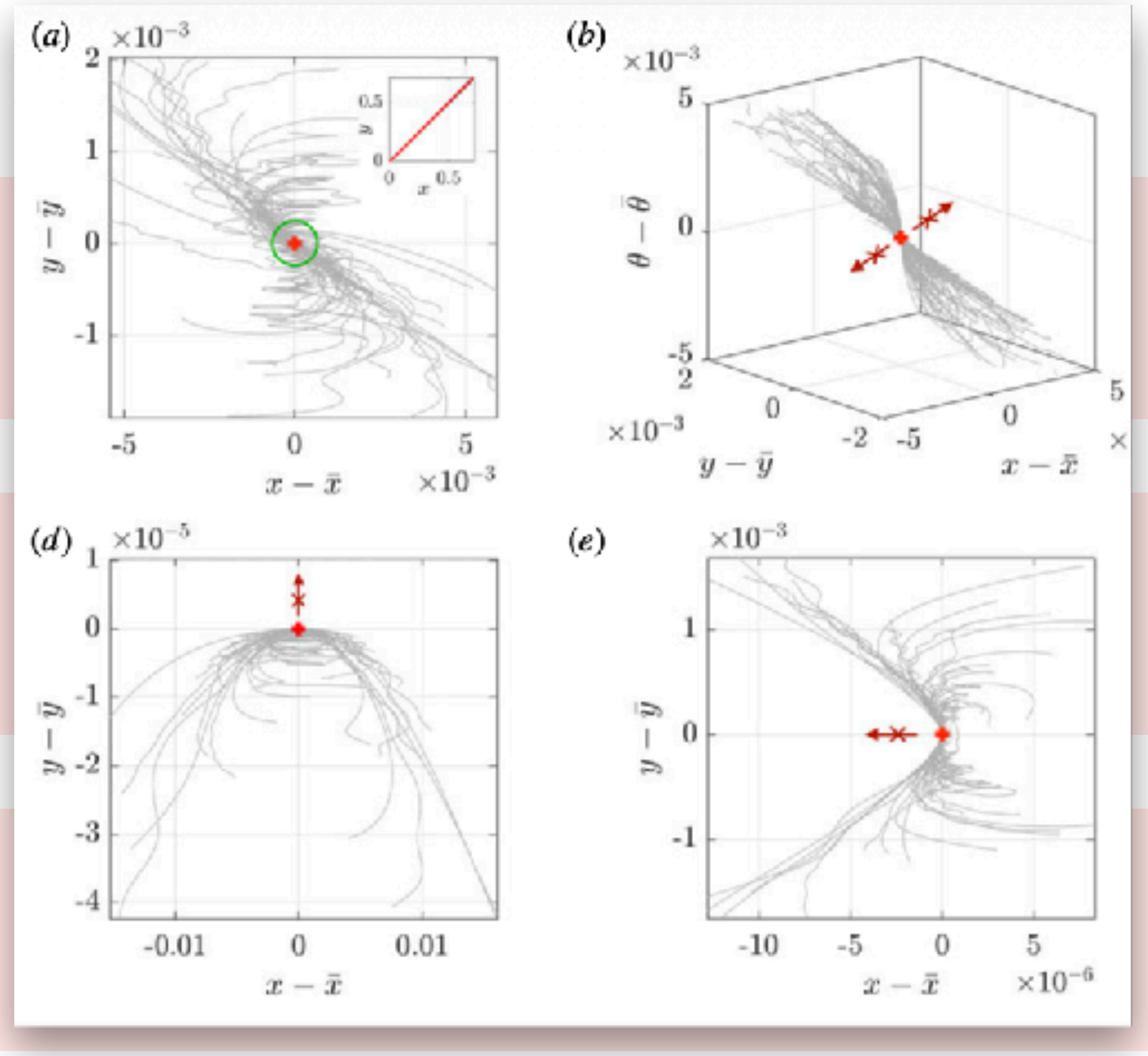
4. Gravity, magnetism

$$\begin{pmatrix} \dot{\mathscr{X}} \\ \dot{\mathscr{S}} \end{pmatrix} = \begin{pmatrix} \mathbf{N}(\mathscr{S}) \\ \mathrm{Id} \end{pmatrix} \dot{\mathscr{S}} + \boxed{\mathbf{F}_{\mathrm{ext}}(\mathscr{X}, \mathscr{S})}$$

Drift

5. Control by deformation

Beyond the canonical swimming model



1.

2.

3.

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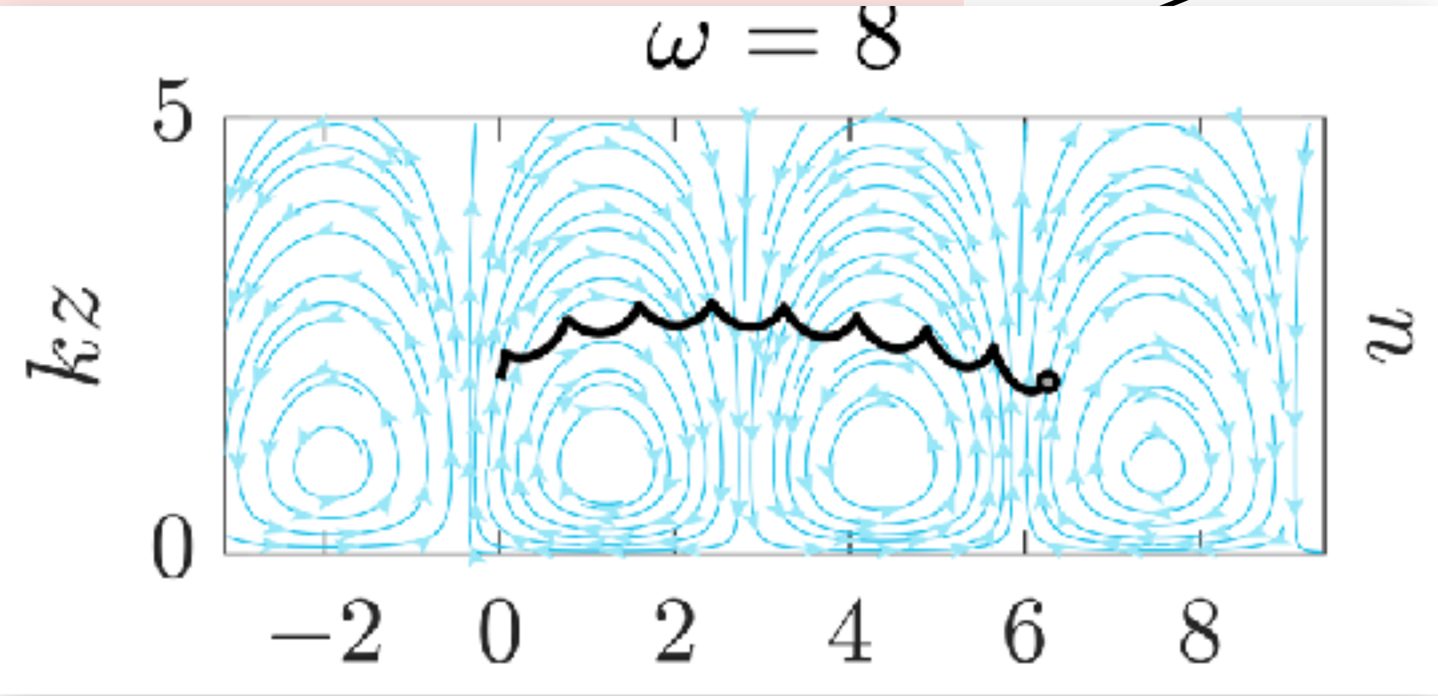
5. Control by c



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Drift

- **Background flow as control**

Beyond the canonical swimming model

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Drift

5. Indirect control

$$\begin{pmatrix} \dot{\mathscr{X}} \\ \dot{\mathscr{S}} \end{pmatrix} = \begin{pmatrix} \mathbf{N}(\mathscr{S})\dot{\mathscr{S}} \\ \mathbf{Q}_0(\mathscr{S}) \end{pmatrix} + \begin{pmatrix} 0 \\ \mathbf{Q}(\mathscr{S}) \end{pmatrix} \mathscr{U}$$

Drift & bigger state

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Memory effect

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Dependence in space

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$$\begin{pmatrix}\dot{\mathscr{X}}\\ \dot{\mathscr{S}}\end{pmatrix} = \begin{pmatrix}\mathbf{N}(\mathscr{S})\\ \mathrm{Id}\end{pmatrix}\dot{\mathscr{S}} +$$

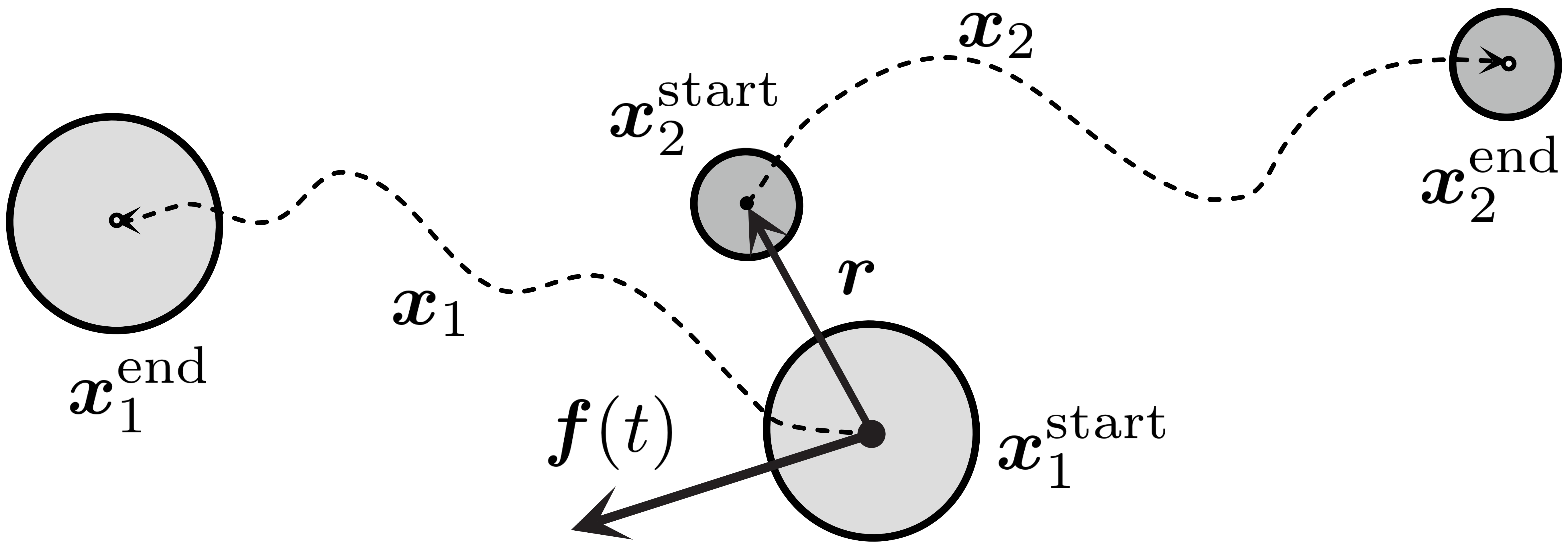
- **Elasticity:** Alouges, DeSimone, Giraldi, Or, Zoppello, etc.
- **Magneto-elastic:** see later

5. Indirect control

$$\begin{pmatrix}\dot{\mathscr{X}}\\ \dot{\mathscr{S}}\end{pmatrix} = \begin{pmatrix}\mathbf{N}(\mathscr{S})\dot{\mathscr{S}}\\ \mathbf{Q}_0(\mathscr{S})\end{pmatrix} + \begin{pmatrix}0\\ \mathbf{Q}(\mathscr{S})\end{pmatrix}\mathscr{U}$$

Drift & bigger state

A case with multiple « swimmers »



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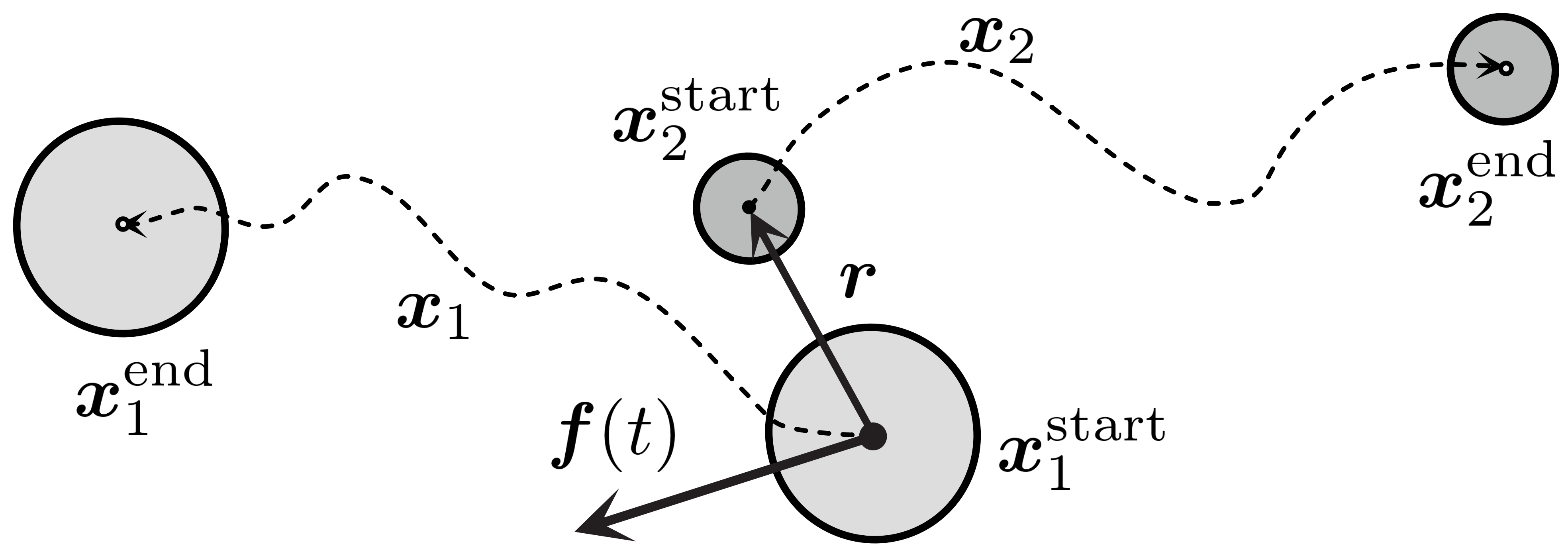


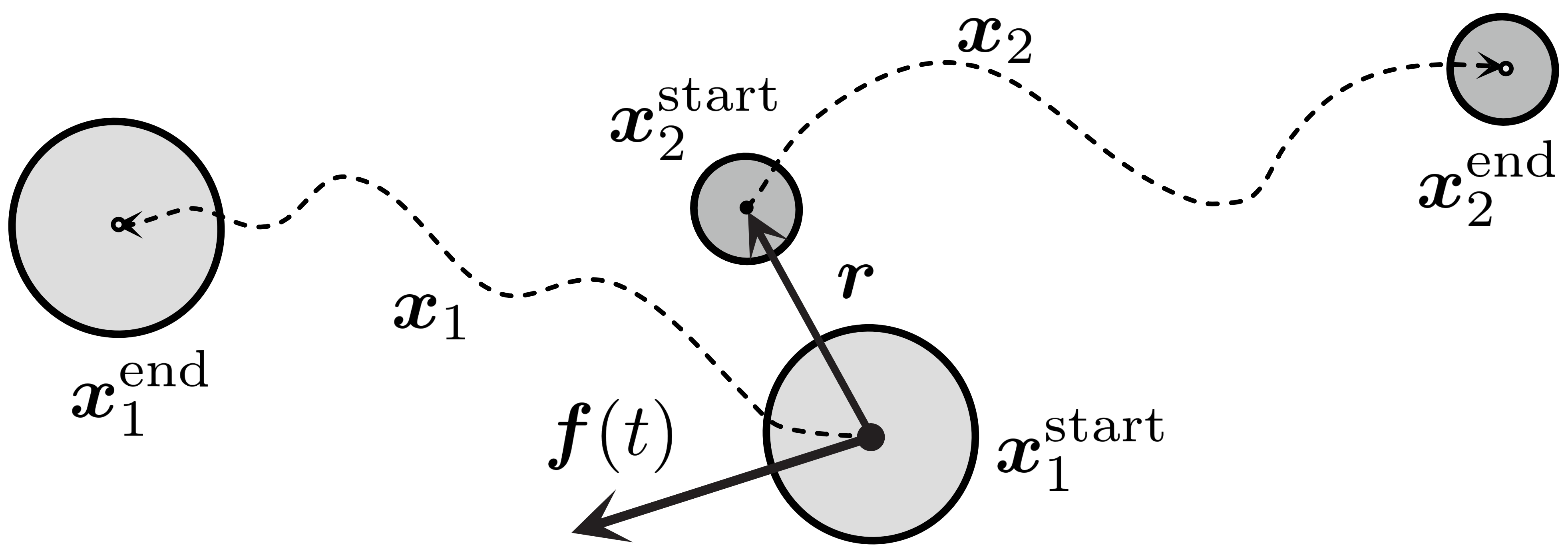
- Motivation: control of **multiple swimmers** / **cargo transport**
- Two spherical particles
- A force $\mathbf{f}$ is applied on one of them
- Control of the position of **both at the same time**

- State : $(\mathbf{x}_1, \mathbf{r})$ (6-dimensional), control : $\mathbf{f} = (f_x, f_y, f_z)$

"Swimming equation" in Stokes flow : $\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix}\mathbf{x}_1\\ \mathbf{r}\end{bmatrix} = \begin{bmatrix} M_{\|}(r)\frac{\mathbf{r}\mathbf{r}^T}{r^2} + M_{\perp}(r)\left(\mathbf{I} - \frac{\mathbf{r}\mathbf{r}^T}{r^2}\right) \\ -m_{\|}(r)\frac{\mathbf{r}\mathbf{r}^T}{r^2} - m_{\perp}(r)\left(\mathbf{I} - \frac{\mathbf{r}\mathbf{r}^T}{r^2}\right)\end{bmatrix}\mathbf{f} = f_x\mathbf{g}_x + f_y\mathbf{g}_y + f_z\mathbf{g}_z$

- **Driftless control system** → Check the LARC for the vector fields $\mathbf{g}_x, \mathbf{g}_\mathbf{y}, \mathbf{g}_\mathbf{z}$

A case with multiple « swimmers »



- Consider the matrix

$$C = \left[\mathbf{g}_x, \mathbf{g}_y, \mathbf{g}_x, [\mathbf{g}_x, \mathbf{g}_y], [\mathbf{g}_x, \mathbf{g}_z], [\mathbf{g}_y, [\mathbf{g}_x, \mathbf{g}_y]]\right]$$

- Its determinant at r is given by

$$\frac{m_\parallel^4 m_\perp^6}{r}\left(\frac{1}{r}\left(\frac{M_\parallel}{m_\parallel} - \frac{M_\perp}{m_\perp}\right) - \left(\frac{M_\perp}{m_\perp}\right)'\right)^3$$

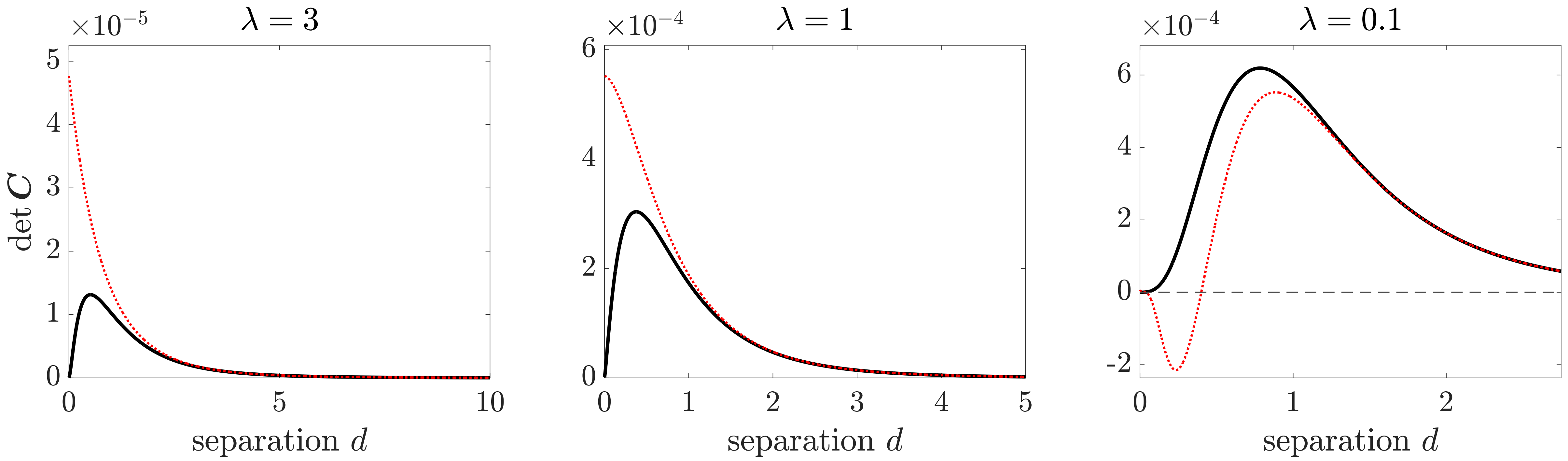
Proposition (Walker, Ishimoto, Gaffney, M., JFM 2022)

- The system is controllable at r if (and only if)

$$\frac{1}{r}\left(\frac{M_\parallel}{m_\parallel} - \frac{M_\perp}{m_\perp}\right) - \left(\frac{M_\perp}{m_\perp}\right)' \neq 0.$$

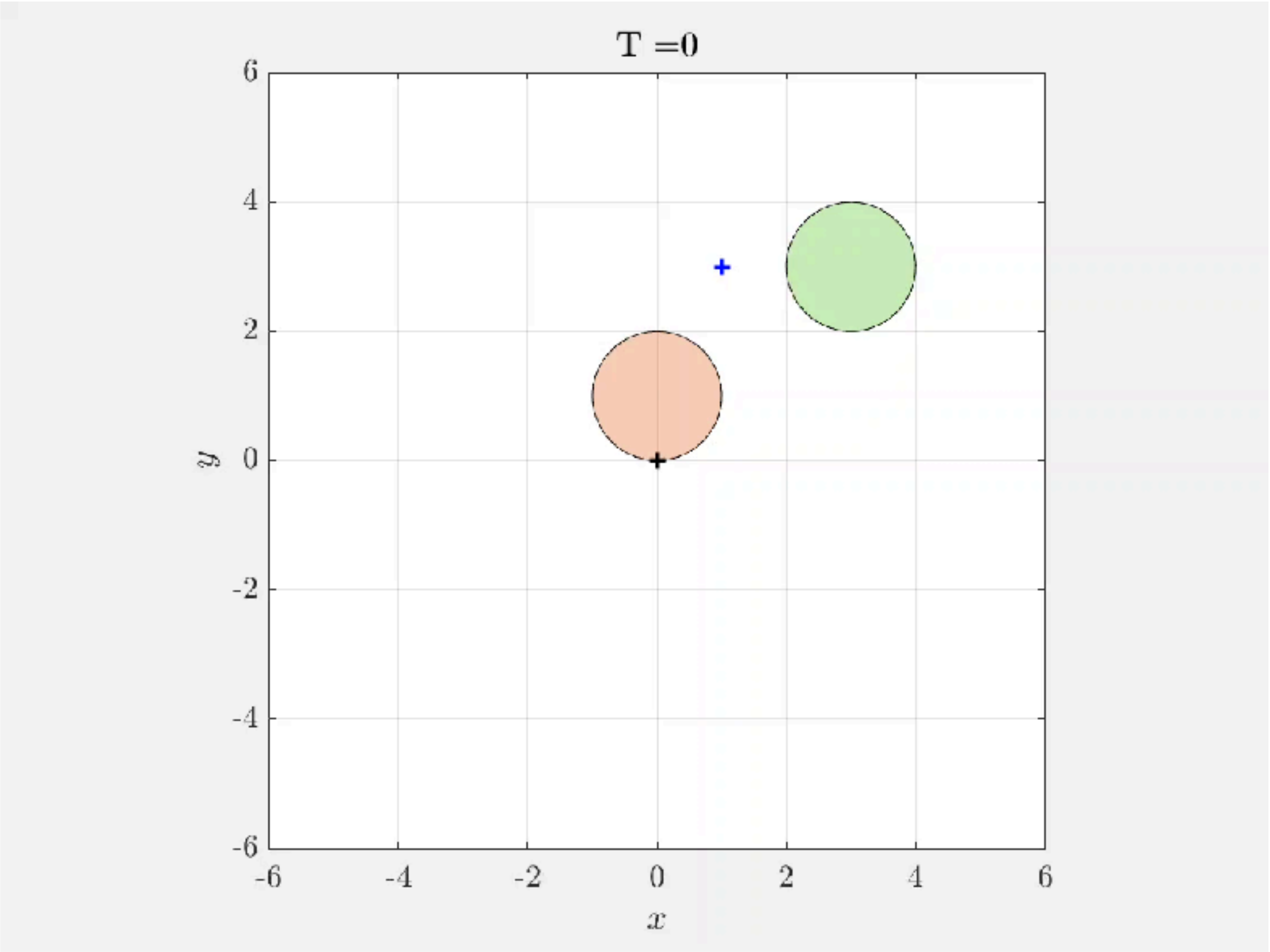
A case with multiple « swimmers »

- The mobility coefficients $M_{\parallel}, m_{\parallel}, M_{\perp}, m_{\perp},$ can be approximated with series in powers of $\dfrac{1}{r}$
- Numerical estimation of the determinant

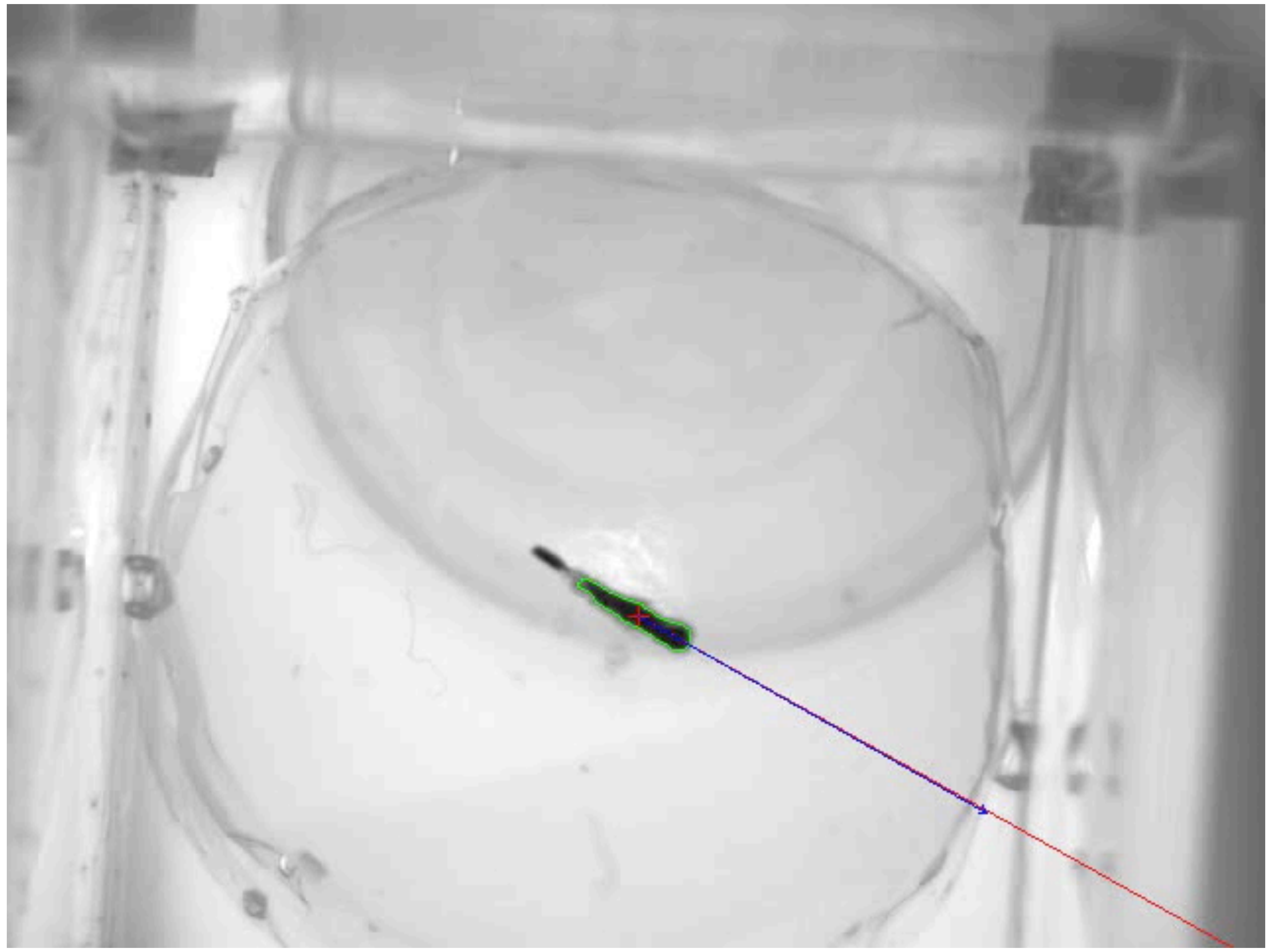


- controllable everywhere!
- more particles...?

- **Motion planning**: explicit build a trajectory, using the Lie brackets as "basis vectors"



Leaving the driftless territory...



El Alaoui-Faris et al. 2019

Conditions for controllability: with a drift

$$(\text{Aff}) : \dot{x} = f_0(x) + \sum_{i=1}^{m} f_i(x) u_i$$

- More complicated when $f_0 \neq 0$: the LARC is only necessary. Counterexample:

$$\dot{x}_1 = x_2^2, \dot{x}_2 = u$$

- **Heuristics:** brackets with an even number of times f_0 are « bad » …

No NS condition for STLC is known!

Scalar-input case

$$\dot{x} = f_0(x) + f_1(x)u$$

- **Sufficient conditions:** Sussmann 1986

$$\text{LARC \& } (\forall k, S_{2k} \subset S_{2k-1})$$

- **Necessary conditions:** Sussmann, Kawski, Krastanov, Beauchard, Marbach, etc.

$$S_2 \not\subset S_1 ?$$

$$[f_1, [f_0, f_1]]$$

Two-control case

$$\dot{x} = f_0(x) + f_1(x)u_1 + f_2(x)u_2$$

- **Sufficient condition**: Sussmann « $S(\theta)$ »
- **Necessary conditions**: none!
  - 2024: **one result coming from a micro-swimmer model**!
  - Update 2025: now additional necessary conditions! (Gherdaoui)

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Theorem (Giraldi, Lissy, M., Pomet, ESAIM:COCV 2024)

Assume $f_2(0) = 0$ and $f_{101}(0) \notin R_1$.

1. If $f_{101}(0) \in R_1 + \mathrm{Span}(f_{121}(0))$ let $\beta \in \mathbb{R}$ such that $f_{101}(0) + \beta f_{121}(0) \in R_1$.
Then, for all $u_{\mathrm{eq},2} \in \mathbb{R}$ such that $u_{\mathrm{eq},2} \neq \beta$, the system is **not STLC** at $(0,(0,u_{\mathrm{eq},2}))$.

2. If $f_{101}(0) \notin R_1 + \mathrm{Span}(f_{121}(0))$, then, for all $u_{\mathrm{eq},2} \in \mathbb{R}$, the system is **not B-STLC** at $(0,(0,u_{\mathrm{eq},2}))$.

- **One may neutralize the bad bracket f_{101} with f_{121}**

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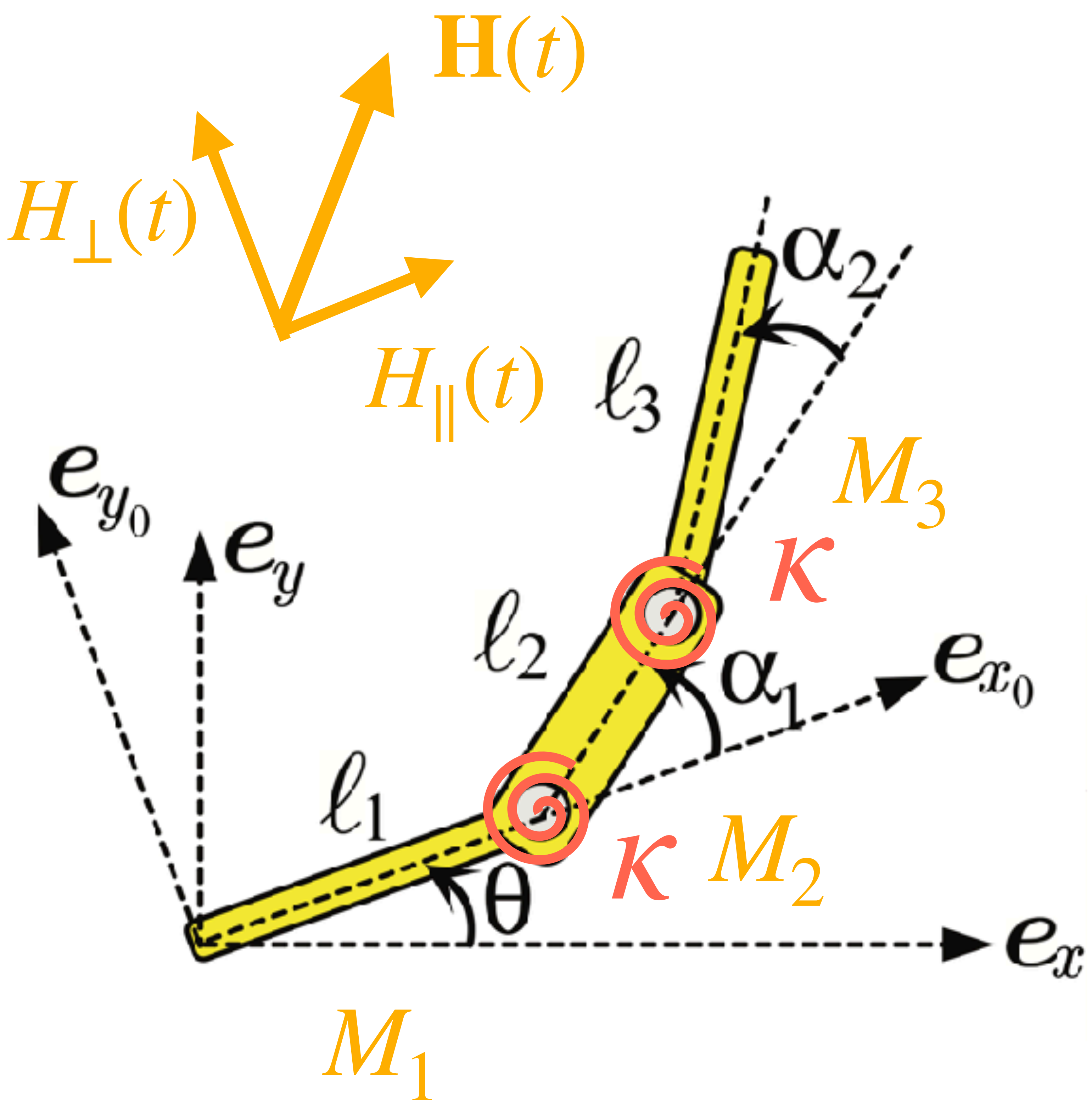
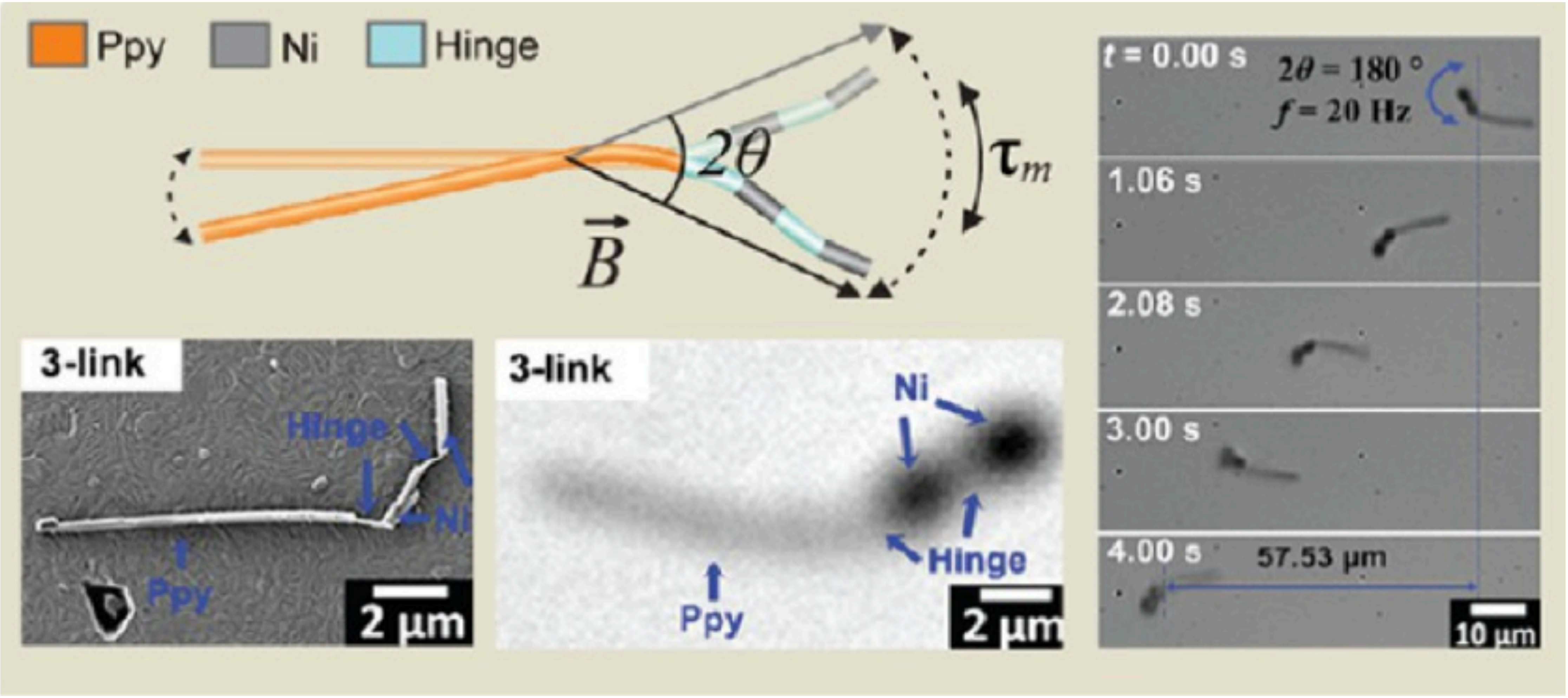
Drift system: magneto-elastic Purcell swimmer





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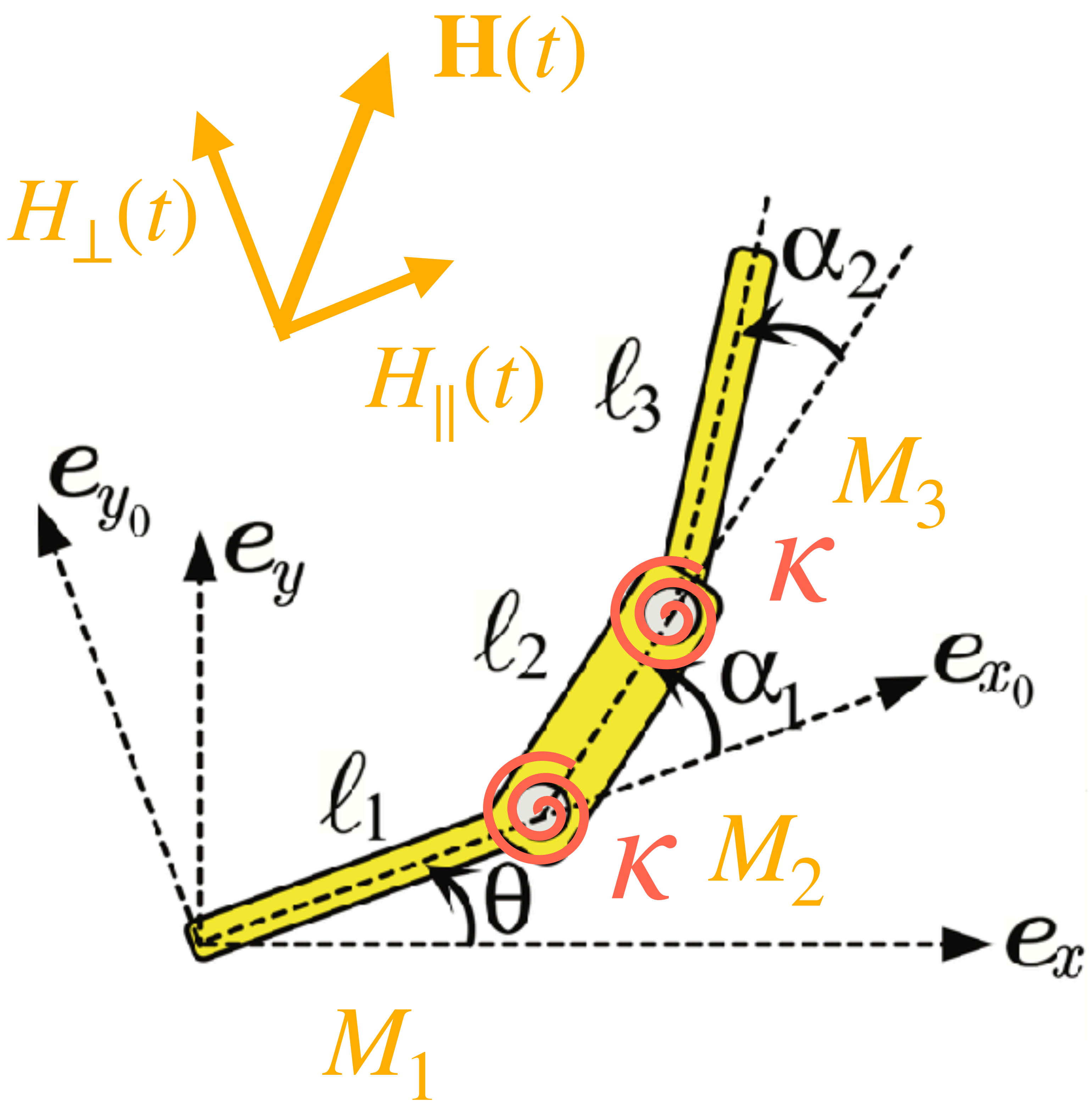
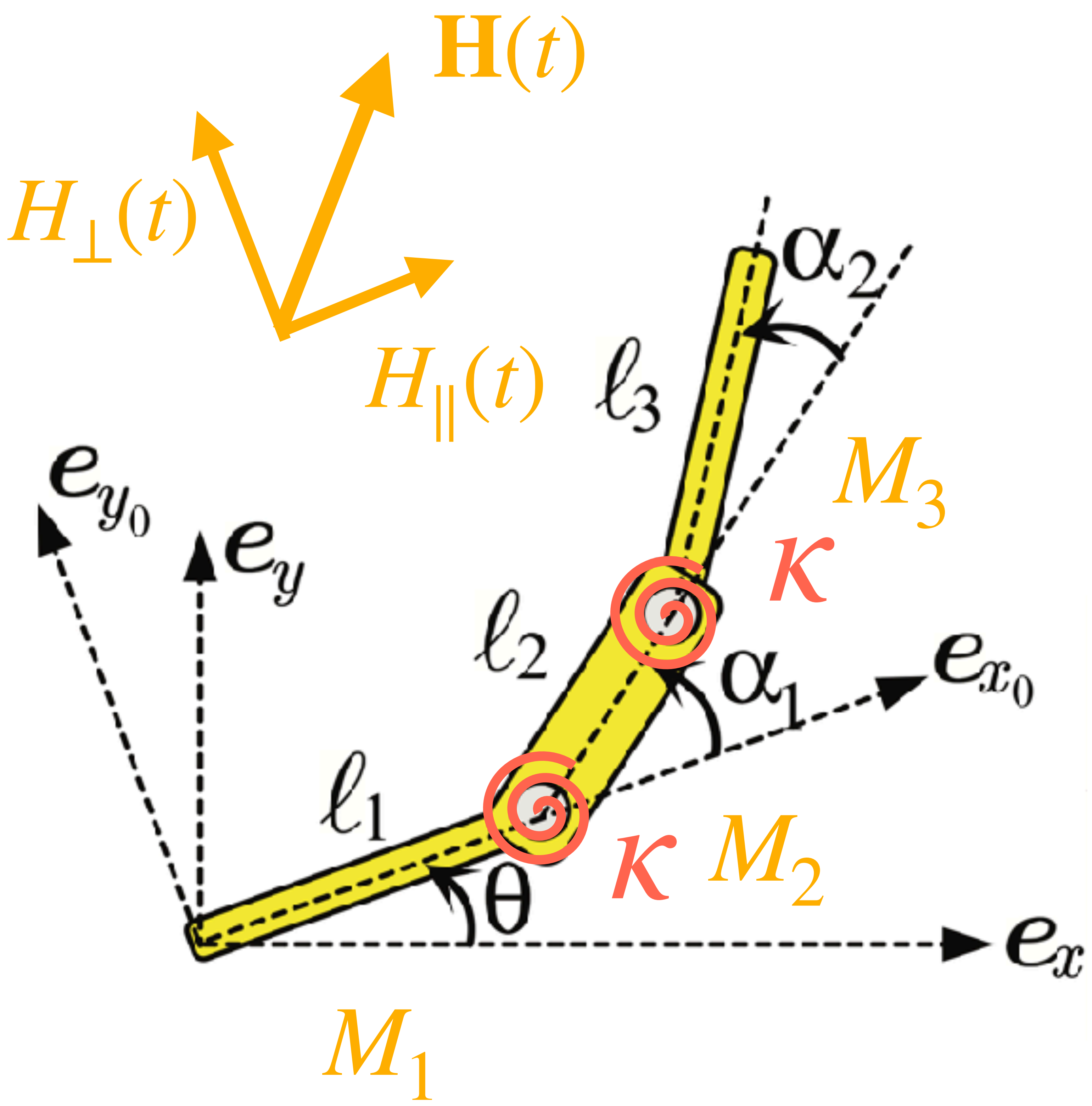


- State: $\mathbf{z} = (x, y, \theta, \alpha_1, \alpha_2)$
- Controls: $(H_\parallel, H_\perp)$

Drift system: magneto-elastic Purcell swimmer

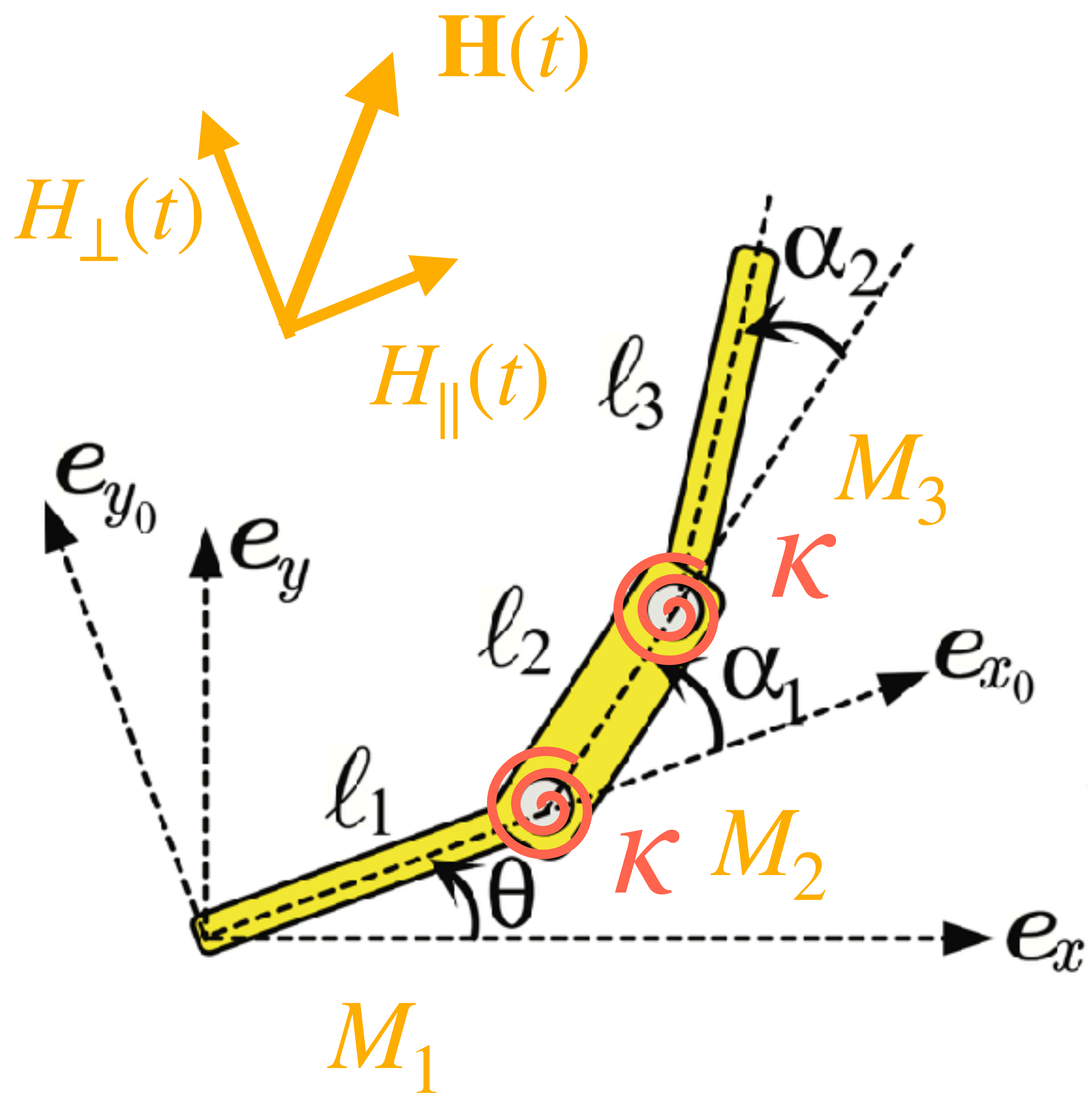


$$\left\{\begin{array}{rcl} \mathbf{f}_1^h + \mathbf{f}_2^h + \mathbf{f}_3^h & = & 0 \\ m_1^h + m_2^h + m_3^h & = & m_1^{mag} + m_2^{mag} + m_3^{mag} \\ m_2^h + m_3^h & = & m_1^{el} + m_2^{mag} + m_3^{mag} \\ m_3^h & = & m_2^{el} + m_3^{mag} \end{array}\right.$$

Equations of motion

- State: $\mathbf{z} = (x, y, \theta, \alpha_1, \alpha_2)$
- Controls: $(H_\parallel, H_\perp)$

Drift system: magneto-elastic Purcell swimmer



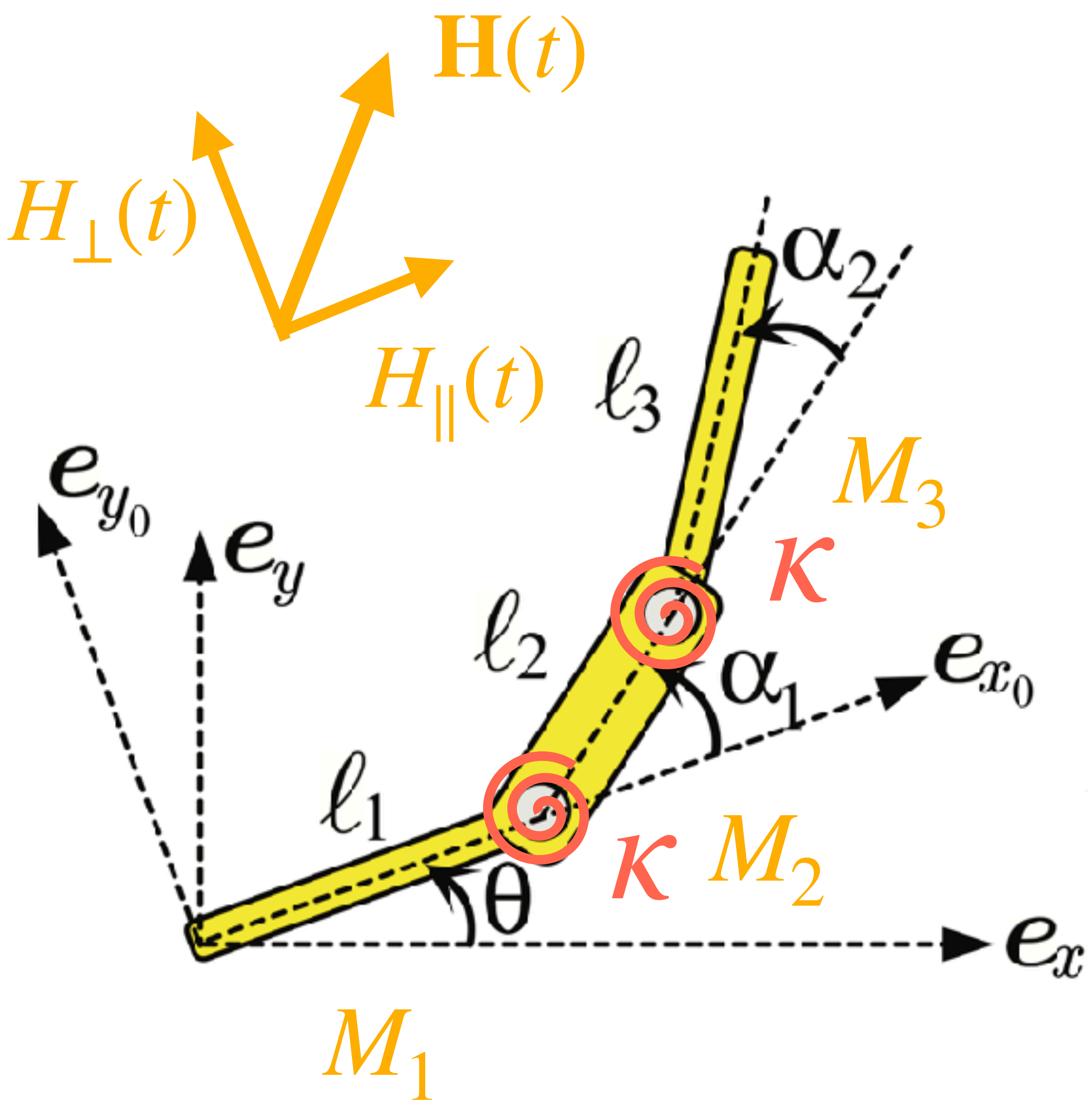
$$
\begin{cases}
\mathbf{f}_1^h + \mathbf{f}_2^h + \mathbf{f}_3^h = 0 \\
m_1^h + m_2^h + m_3^h = m_1^{mag} + m_2^{mag} + m_3^{mag} \\
m_2^h + m_3^h = m_1^{el} + m_2^{mag} + m_3^{mag} \\
m_3^h = m_2^{el} + m_3^{mag}
\end{cases}
$$

Hydrodynamics Elasticity Magnetism

Equations of motion

- State: $\mathbf{z} = (x, y, \theta, \alpha_1, \alpha_2)$
- Controls: $(H_\parallel, H_\perp)$

Drift system: magneto-elastic Purcell swimmer



$$\begin{cases} \mathbf{f}_1^h + \mathbf{f}_2^h + \mathbf{f}_3^h = 0 \\ m_1^h + m_2^h + m_3^h = m_1^{mag} + m_2^{mag} + m_3^{mag} \\ m_2^h + m_3^h = m_1^{el} + m_2^{mag} + m_3^{mag} \\ m_3^h = m_2^{el} + m_3^{mag} \end{cases}$$

$$m_i^{\mathrm{mag}} = M_i \mathbf{H} \times \mathbf{e}_{i,\parallel}$$

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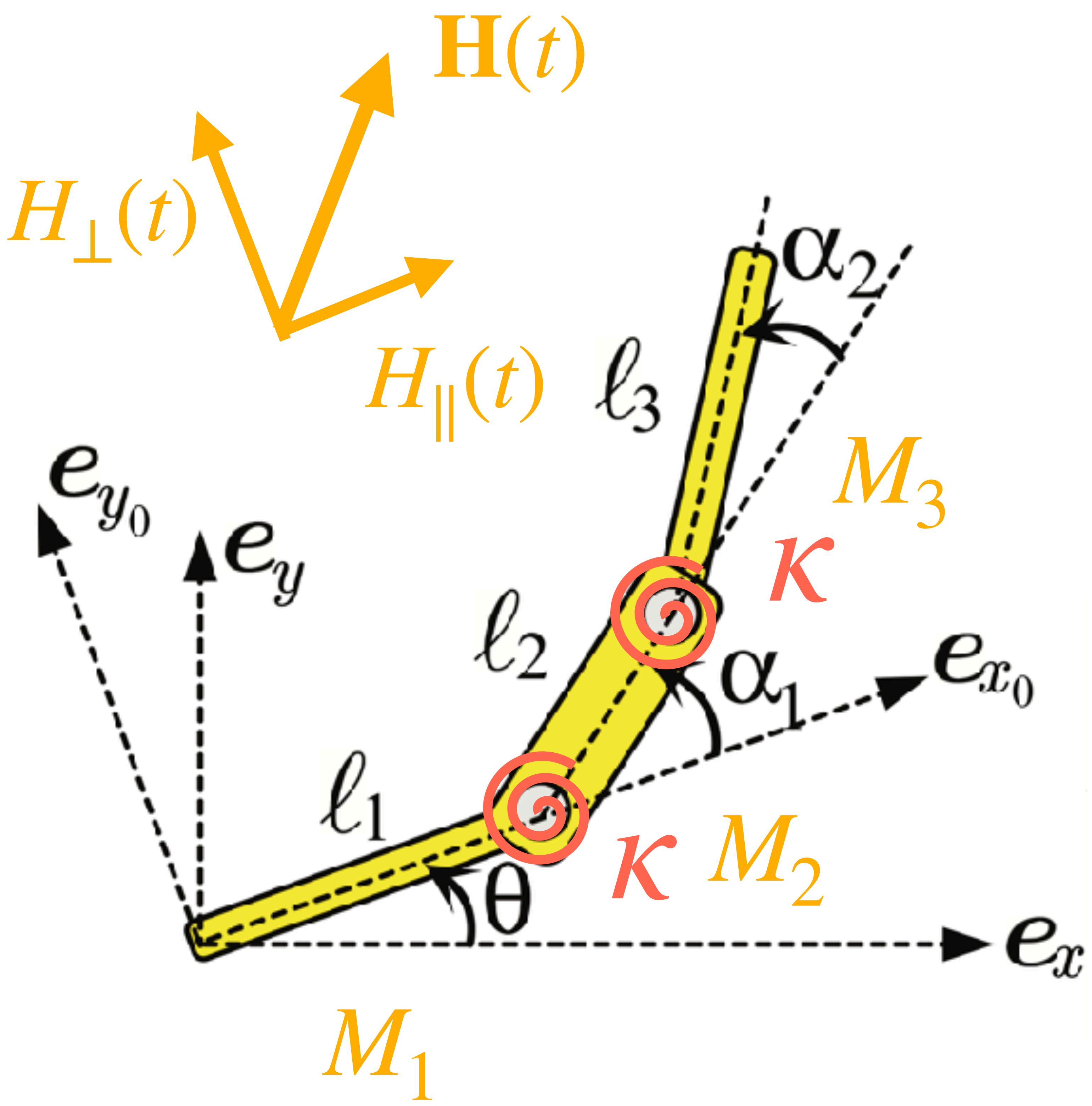

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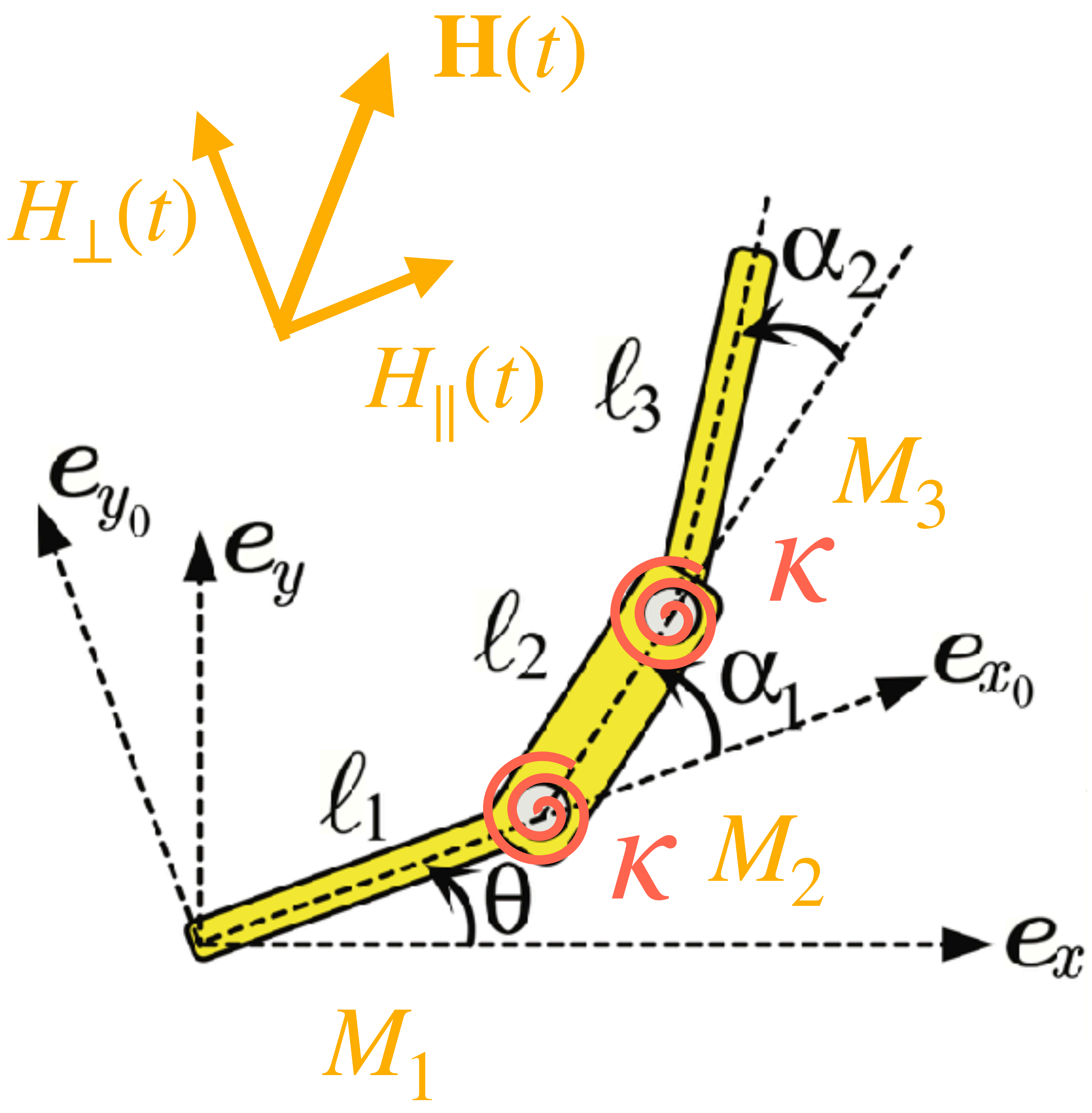
- State: $\mathbf{z} = (x, y, \theta, \alpha_1, \alpha_2)$
- Controls: $(H_\parallel, H_\perp)$
- Equilibrium points: $\alpha_1 = \alpha_2 = 0, H_\perp = 0, H_\parallel$ **arbitrary**

Drift system: magneto-elastic Purcell swimmer



$$\begin{cases} \mathbf{f}_1^h + \mathbf{f}_2^h + \mathbf{f}_3^h & = & 0 \\ m_1^h + m_2^h + m_3^h & = & m_1^{mag} + m_2^{mag} + m_3^{mag} \\ m_2^h + m_3^h & = & m_1^{el} + m_2^{mag} + m_3^{mag} \\ m_3^h & = & m_2^{el} + m_3^{mag} \end{cases}$$

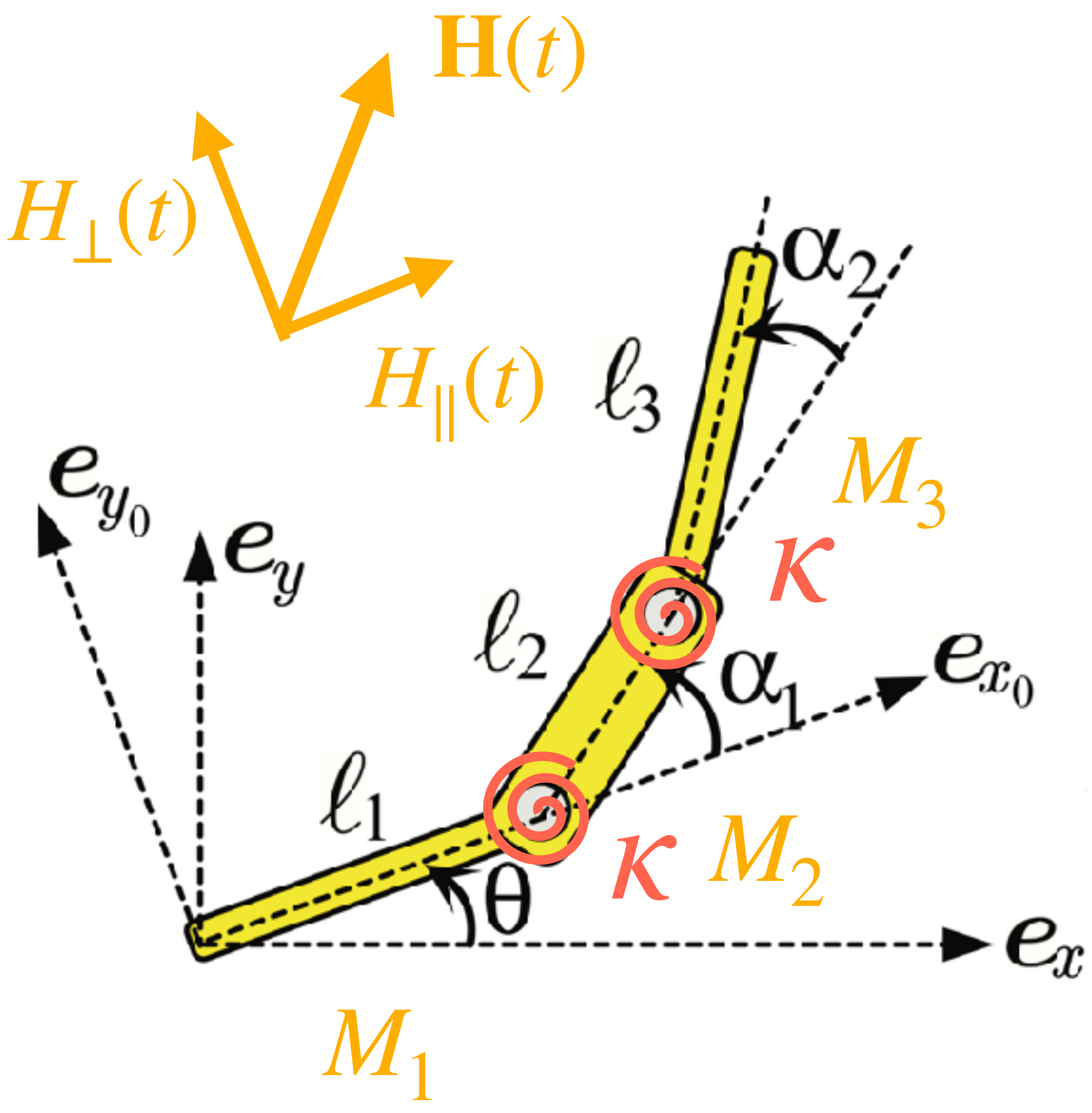
$$m_i^{\text{mag}} = M_i \mathbf{H} \times \mathbf{e}_{i,\parallel}$$

$$\dot{\mathbf{z}} = \mathbf{F}_0(\alpha_1, \alpha_2) + \mathbf{F}_\parallel(\alpha_1, \alpha_2) H_\parallel(t) + \mathbf{F}_\perp(\alpha_1, \alpha_2) H_\perp(t)$$

Control-affine system with a drift

- State: $\mathbf{z} = (x, y, \theta, \alpha_1, \alpha_2)$
- Controls: $(H_\parallel, H_\perp)$
- Equilibrium points: $\alpha_1 = \alpha_2 = 0, H_\perp = 0,$ $H_\parallel$ **arbitrary**

Drift system: magneto-elastic Purcell swimmer



$$\begin{cases} \mathbf{f}_1^h + \mathbf{f}_2^h + \mathbf{f}_3^h = 0 \\ m_1^h + m_2^h + m_3^h = m_1^{mag} + m_2^{mag} + m_3^{mag} \\ m_2^h + m_3^h = m_1^{el} + m_2^{mag} + m_3^{mag} \\ m_3^h = m_2^{el} + m_3^{mag} \end{cases}$$

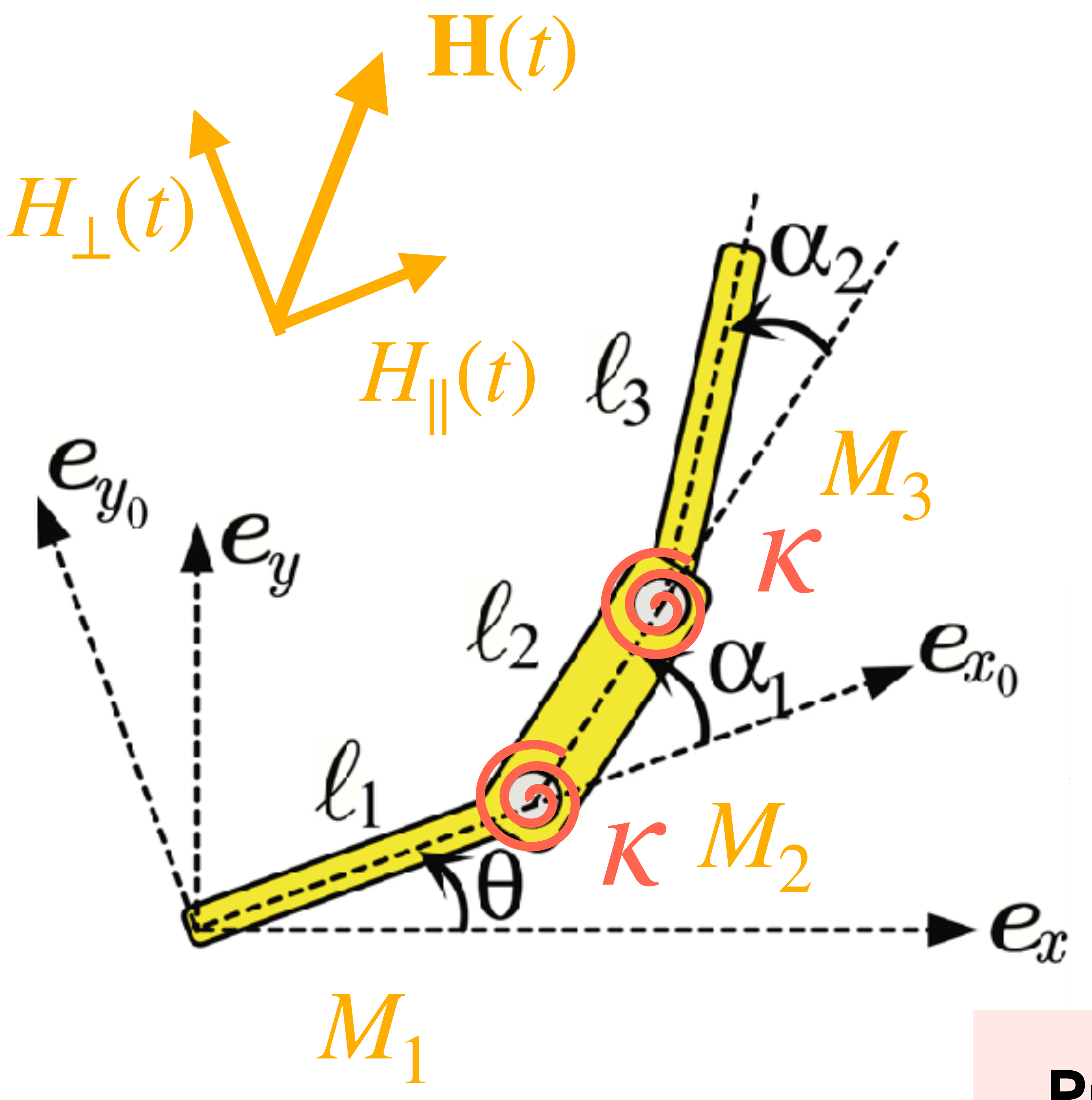
$$m_i^{\mathrm{mag}} = M_i \mathbf{H} \times \mathbf{e}_{i,\parallel}$$

$$\dot{\mathbf{z}} = \mathbf{F}_0(\alpha_1, \alpha_2) + \mathbf{F}_\parallel(\alpha_1, \alpha_2) H_\parallel(t) + \mathbf{F}_\perp(\alpha_1, \alpha_2) H_\perp(t)$$

- State: $\mathbf{z} = (x, y, \theta, \alpha_1, \alpha_2)$
- Controls: $(H_\parallel, H_\perp)$
- Equilibrium points: $\alpha_1 = \alpha_2 = 0, H_\perp = 0,$ $H_\parallel$ **arbitrary**

Around which points (= **which value of** $H_\parallel$)
do we have local controllabilty?

Drift system: magneto-elastic Purcell swimmer



$$\begin{cases} \mathbf{f}_1^h + \mathbf{f}_2^h + \mathbf{f}_3^h = 0 \\ m_1^h + m_2^h + m_3^h = m_1^{mag} + m_2^{mag} + m_3^{mag} \\ m_2^h + m_3^h = m_1^{el} + m_2^{mag} + m_3^{mag} \\ m_3^h = m_2^{el} + m_3^{mag} \end{cases}$$

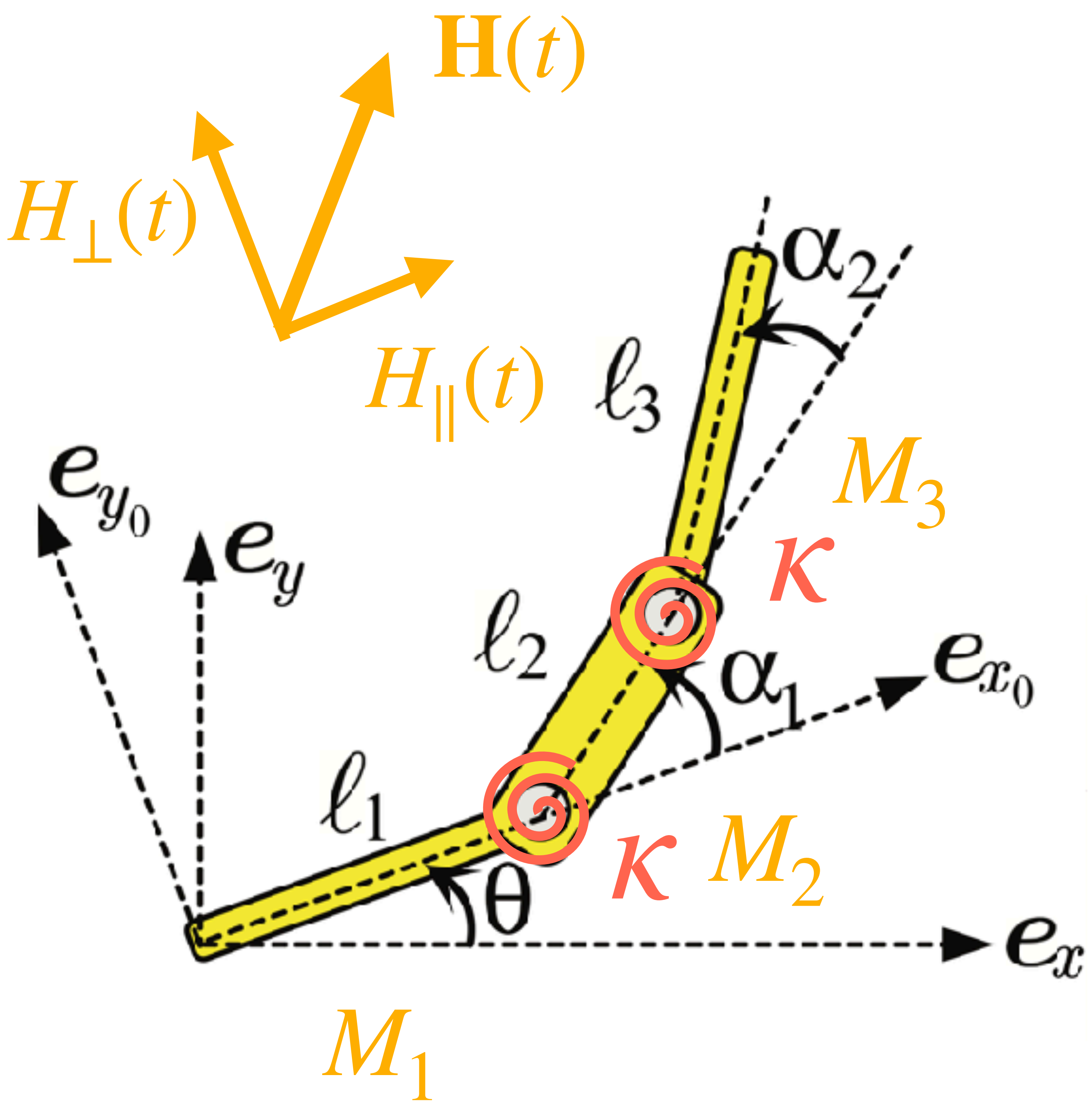
$$\dot{\mathbf{z}} = \mathbf{F}_0(\alpha_1,\alpha_2) + \mathbf{F}_\parallel(\alpha_1,\alpha_2)H_\parallel(t) + \mathbf{F}_\perp(\alpha_1,\alpha_2)H_\perp(t)$$

Proposition: The swimmer is locally controllable if and only if

$$H_\parallel = \frac{[\mathbf{F}_\perp,[\mathbf{F}_0,\mathbf{F}_\perp]](0,0)}{[\mathbf{F}_\perp,[\mathbf{F}_\parallel,\mathbf{F}_\perp]](0,0)}.$$

Moreau, IEEE Control Syst. Lett. 2019

Drift system: magneto-elastic Purcell swimmer



$$\begin{cases} \mathbf{f}_1^h + \mathbf{f}_2^h + \mathbf{f}_3^h = 0 \\ m_1^h + m_2^h + m_3^h = m_1^{mag} + m_2^{mag} + m_3^{mag} \\ m_2^h + m_3^h = m_1^{el} + m_2^{mag} + m_3^{mag} \\ m_3^h = m_2^{el} + m_3^{mag} \end{cases}$$

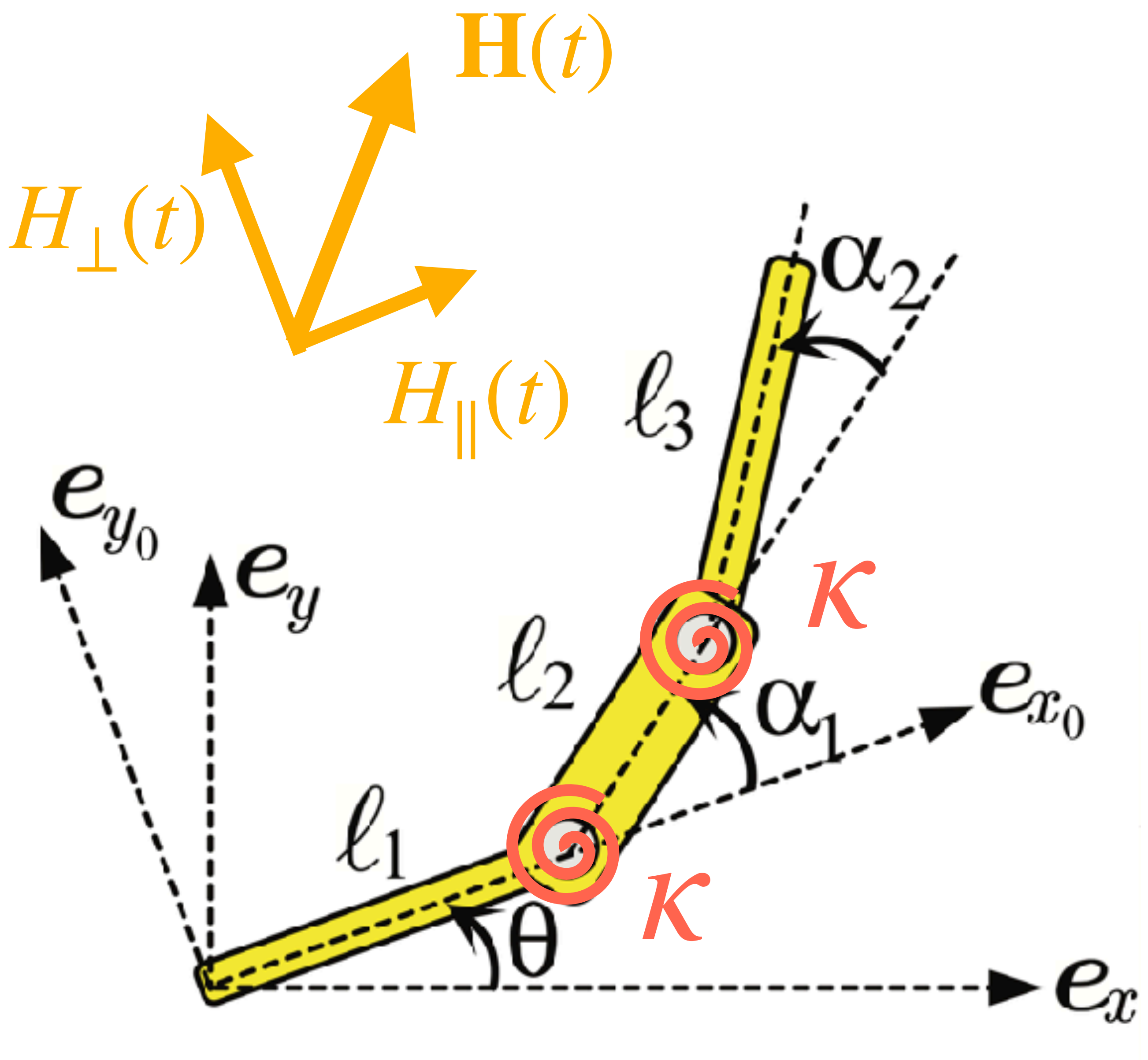
$$\dot{\mathbf{z}} = \mathbf{F}_0(\alpha_1, \alpha_2) + \mathbf{F}_\parallel(\alpha_1, \alpha_2) H_\parallel(t) + \mathbf{F}_\perp(\alpha_1, \alpha_2) H_\perp(t)$$

Proposition: The swimmer is STLC if and only if

$$H_\parallel = \frac{[\mathbf{F}_\perp, [\mathbf{F}_0, \mathbf{F}_\perp]](0,0)}{[\mathbf{F}_\perp, [\mathbf{F}_\parallel, \mathbf{F}_\perp]](0,0)} = \kappa \frac{17(M_1 + M_3) - 16M_2}{-7M_2^2 + 9M_2(M_1 + M_3) + 5M_1M_3}.$$

Moreau, IEEE Control Syst. Lett. 2019

Drift system: magneto-elastic Purcell swimmer

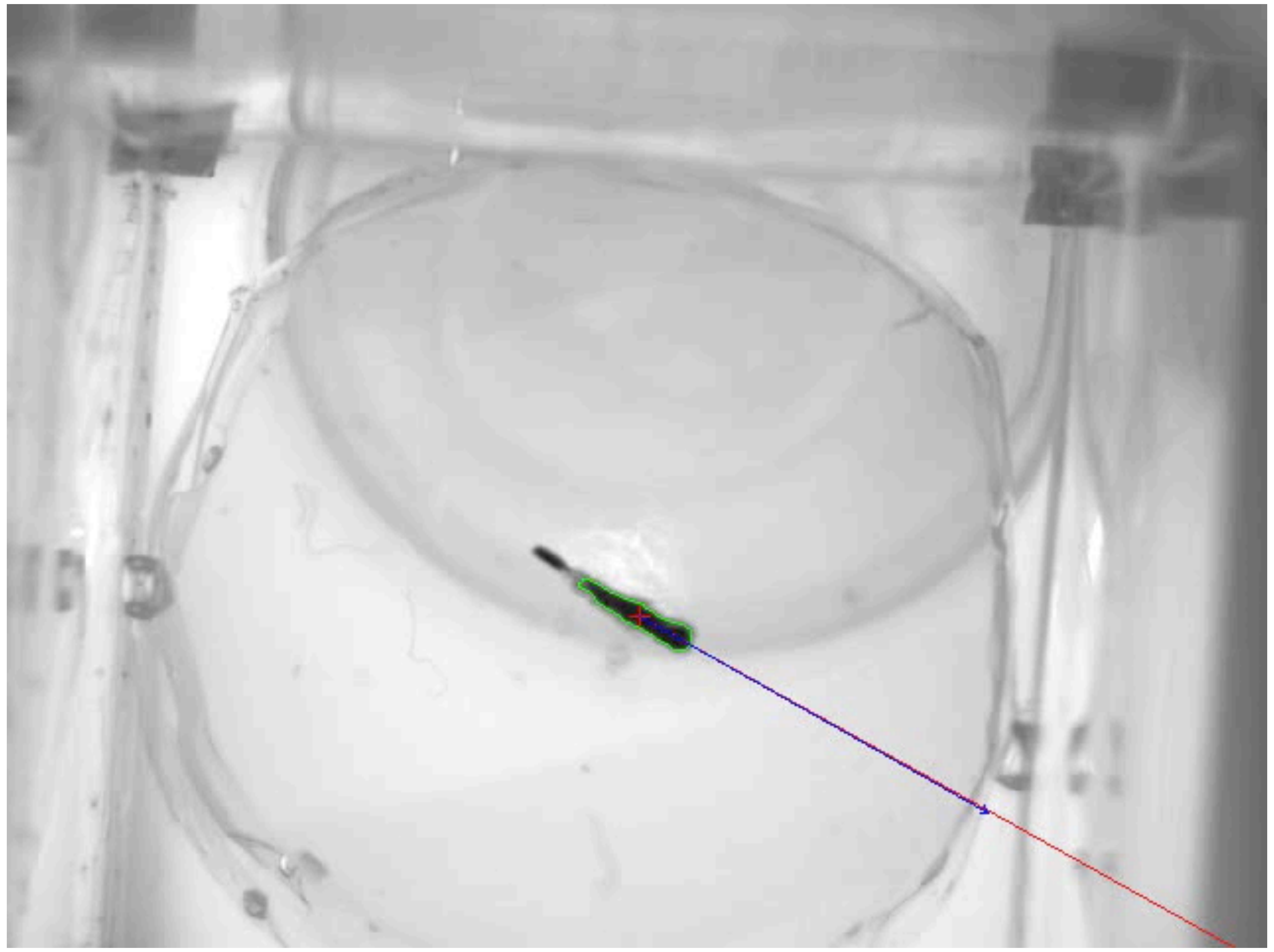


Proposition: The swimmer is STLC if and only if

$$H_\parallel = \frac{[\mathbf{F}_\perp, [\mathbf{F}_0, \mathbf{F}_\perp]](0,0)}{[\mathbf{F}_\perp, [\mathbf{F}_\parallel, \mathbf{F}_\perp]](0,0)} = \kappa \frac{17(M_1 + M_3) - 16M_2}{-7M_2^2 + 9M_2(M_1 + M_3) + 5M_1M_3}.$$

"The control $H_\parallel$ must be close to this value in order to neutralise the drift"

Mystery solved



El Alaoui-Faris et al. 2019

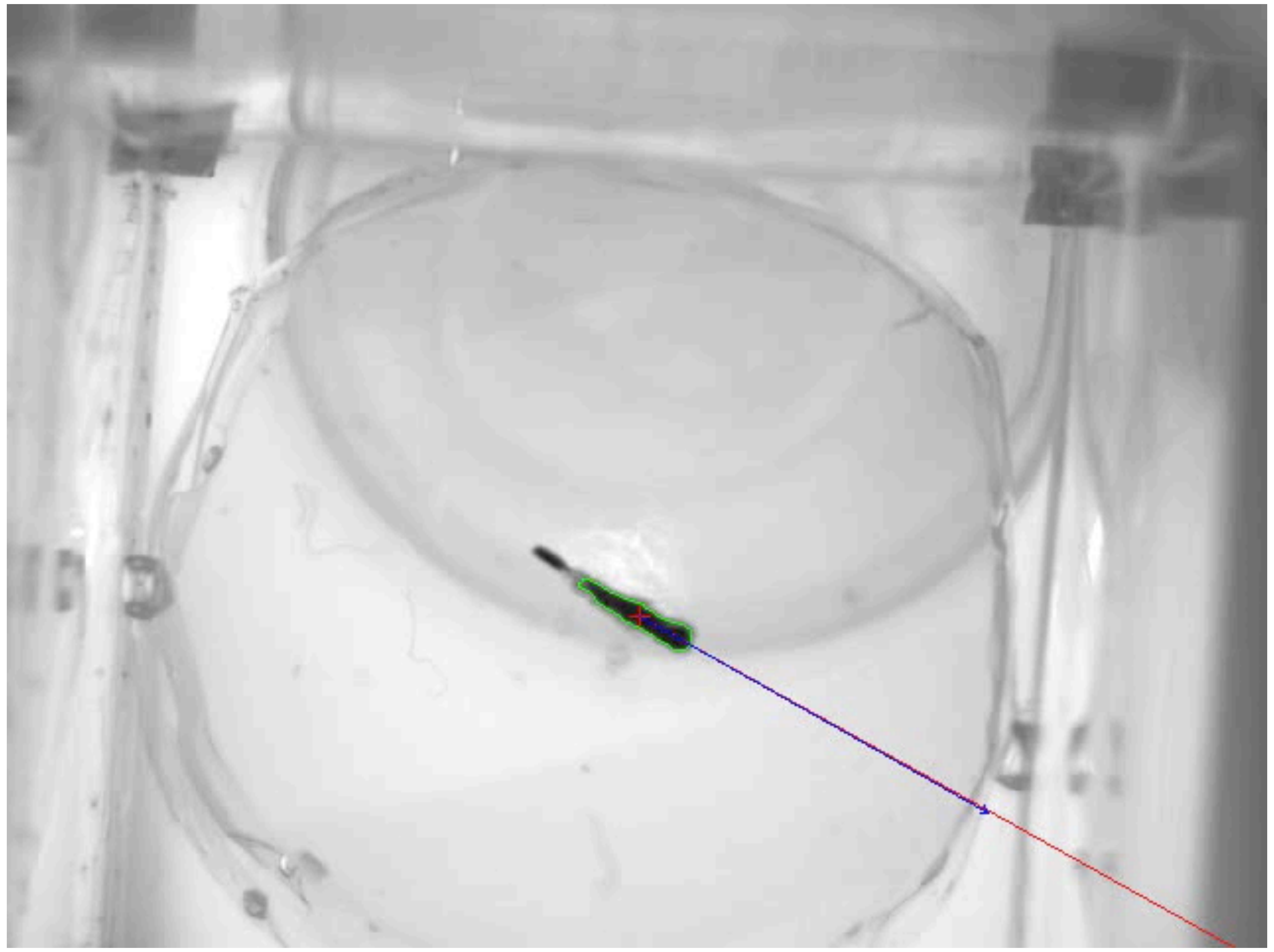
Magnetic field: vertical oscillating
+ constant horizontal

$H_{\|} = -0.2$

$H_{\|} = -0.65$

$H_{\|} = -1$

Mystery solved



El Alaoui-Faris et al. 2019

Magnetic field: vertical oscillating
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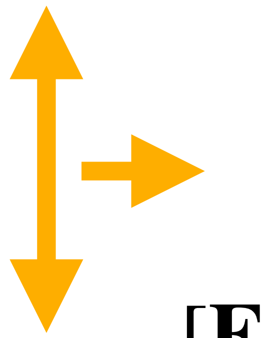
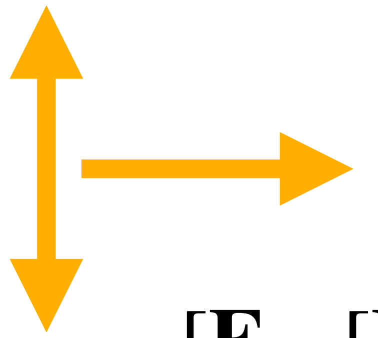


$$H_{\|} < \frac{[\mathbf{F}_{\perp}, [\mathbf{F}_0, \mathbf{F}_{\perp}]](0,0)}{[\mathbf{F}_{\perp}, [\mathbf{F}_{\|}, \mathbf{F}_{\perp}]](0,0)}$$



$$H_{\|} = \frac{[\mathbf{F}_{\perp}, [\mathbf{F}_0, \mathbf{F}_{\perp}]](0,0)}{[\mathbf{F}_{\perp}, [\mathbf{F}_{\|}, \mathbf{F}_{\perp}]](0,0)}$$

$$H_{\|} > \frac{[\mathbf{F}_{\perp}, [\mathbf{F}_0, \mathbf{F}_{\perp}]](0,0)}{[\mathbf{F}_{\perp}, [\mathbf{F}_{\|}, \mathbf{F}_{\perp}]](0,0)}$$

Two-control case, higher-order brackets

- **seven other brackets** of order 5 are involved, via the map

$$D_{u_2^{eq}}(\lambda_1,\lambda_2) = \lambda_1^2(f_{01,001}(0) - u_2^{eq} f_{01,201}(0)) + \lambda_2^2(f_{21,021}(0) - u_2^{eq} f_{21,221}(0)) \\ -\lambda_1\lambda_2(f_{21,001}(0) + f_{01,021}(0) - u_2^{eq}(f_{21,201}(0) + f_{01,221}(0))),$$

Theorem

Assume that $f_{101}(0)$, $f_{121}(0)$ and $f_{21,01}(0)$ belong to R_1 and let $u_2^{eq} \in \mathbb{R}$. Then :

1. *If there exists a linear form φ on $\mathbb{R}^n$ such that*

$$R_1 \subset \ker \varphi, \tag{12}$$

$$(\lambda_1,\lambda_2) \mapsto \langle \varphi\,,\, D_{u_2^{eq}}(\lambda_1,\lambda_2)\rangle \text{ is positive definite,} \tag{13}$$

   then system (S_2) is not $(\mathrm{W}^{1,\infty},\mathrm{L}^\infty)$-STLC at $(0,(0,u_2^{eq}))$.

2. *If there exists a linear form ψ on $\mathbb{R}^n$ such that*

$$R' \subset \ker \psi, \tag{14}$$

$$(\lambda_1,\lambda_2) \mapsto \langle \psi\,,\, D_{u_2^{eq}}(\lambda_1,\lambda_2)\rangle \text{ is positive definite,} \tag{15}$$

   then, system (S_2) is not $(\mathrm{W}^{1,\infty}, B)$-STLC at $(0,(0,u_2^{eq}))$.

- structurally similar to the one for f_{101} but more technical

$$[[f_0,f_1],[f_0,[f_0,f_1]]]$$

$$[[f_2,f_1],[f_0,[f_0,f_1]]]$$

$$\cdots$$

$$[[f_2,f_1],[f_2,[f_2,f_1]]]$$

Summary

- Overview of control theory applied to microswimming
- Future challenges:
  - More controllability results for systems with a drift?
  - Multiple swimmers problem : controllability for more particles
  - “Break more assumptions” from the canonical swimming equation

References

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The control of particles in the Stokes limit
B.J. Walker[1,†], K. Ishimoto[2], E.A. Gaffney[1] and C. Moreau[2]

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https://doi.org/10.7566/JPSJ.92.121005
Special Topics
Advances in the Physics of Biofluid Locomotion
Controllability and Optimal Control of Microswimmers: Theory and Applications
Clément Moreau*
RIMS, Kyoto University

ESAIM: COCV 30 (2024) 4
https://doi.org/10.1051/cocv/2023073
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NECESSARY CONDITIONS FOR LOCAL CONTROLLABILITY OF A PARTICULAR CLASS OF SYSTEMS WITH TWO SCALAR CONTROLS
Laetitia Giraldi[1], Pierre Lissy[2], Clément Moreau[1,2,3*] and Jean-Baptiste Pomet[1]

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Local Controllability of a Magnetized Purcell’s Swimmer
Clément Moreau

Thank you for your attention!