Optimal control of sweeping processes: theoretical framework and numerical approximation

Francisco J. Silva (XLIM, Université de Limoges)

Based on an oingoing work with Justina Gianatti (Universidad de Rosario) and Emilio Vilches (Universidad de O'Higgins).

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Plan

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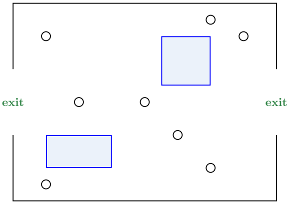
Crowd motion [5][1]

Let us consider a model of crowd motion in emergency evacuations:



In this model:

- Consider N persons identified as nonoverlapping rigid disks in $\mathbb{R}^2$.
- Each individual has a spontaneous velocity that he would like to have in absence of other people.

[1] [5] B. Maury and J. Venel: A discrete contact model for crowd motion. ESAIM Math. Model. Numer. Anal. (2011)

The vector of positions $q = (q_1, \dots, q_N) \in \mathbb{R}^{2N}$ has to belong to the set

$$Q := \left\{ q \in \mathbb{R}^{2N} \;\middle|\; \forall i \neq j,\ D_{ij}(q) \geq 0 \right\},$$

where $D_{ij}(q) := \|q_i - q_j\| - 2r$ is the distance between the disk i and j.
If the global spontaneous velocity of the crowd is denoted by

$$V(t, q) = (V_1(t, q_1), \dots, V_N(t, q_N)) \in \mathbb{R}^{2N},$$

the crowd motion can be described by the following projected differential equation:

$$\frac{dq}{dt} = \mathrm{proj}_{\mathcal{T}_Q(q)}(V(t, q)),$$

where

$$\mathcal{T}_Q(q) = \left\{ v \in \mathbb{R}^{2N} \;\middle|\; \forall i < j,\ D_{ij}(q) = 0 \Rightarrow \nabla D_{ij}(q) \cdot v \geq 0 \right\},$$

is the set of admissible velocities.
The last projected differential equation is equivalent to:

$$\frac{dq}{dt} \in -\mathcal{N}_Q(q) + V(t, q).$$

Moreau's sweeping process

In the previous example, the motion can be described by the so-called *sweeeping processes*:

$$\begin{cases} \dot{\gamma}(t) \in -\mathcal{N}_{\mathbf{C}(t)}(\gamma(t)) + f(t,\gamma(t)) & \text{a.e. } t \in [0,T], \\ \gamma(0) = x \in \mathbf{C}(0), \end{cases}$$

where $\mathbf{C}(t)$ is a closed set for all $t \in [0,T]$ and $f\colon [0,T] \times \mathbb{R}^d \to \mathbb{R}^d$.

- The sweeping process was introduced by J.-J. Moreau in 1971[2] to model an elastoplastic mechanical system.
- It appears in several fields such as nonsmooth electrical circuits, nonsmooth mechanics, crowd motion, hysteresis phenomena, etc.

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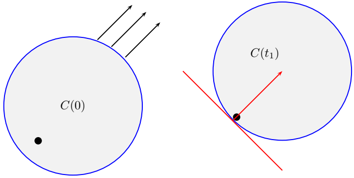
2 [6] J.-J. Moreau: Rafle par un convexe variable I. Sém. Anal. Convexe Montpellier (1971), Exposé 15.

Interpretation of the sweeping process

Consider a large ring that contains a small ball. The ring will start to move at time $t = 0$.
Depending on the motion of the ring, the ball will just stay where it is (in case it is not hit by the ring), or otherwise it is swept by the ring.



Interpretation of the sweeping process

Mathematically,

$$\begin{cases} \dot{\gamma}(t) \in -\mathcal{N}_{\mathbf{C}(t)}\left(\gamma(t)\right) \ \text{ a.e. } \ t \in [0,T], \\ \gamma(0) = x \in \mathbf{C}(0), \end{cases} \tag{1}$$

where

- $\gamma(t)$ is the position of the ball at time t.
- $\mathbf{C}(t)$ is the moving set (the ring and its interior).
- $N_{\mathbf{C}(t)}\left(\gamma(t)\right)$ is some appropriate outward normal cone of $\mathbf{C}(t)$ at $\gamma(t) \in \mathbf{C}(t)$.

In the general setting, the set $\mathbf{C}(t)$ is allowed to change its shape while is moving.

First basic existence result

Theorem (Moreau 1971 [6, 7])

If the sets $(\mathbf{C}(t))_{t\geq 0}$ *are nonempty, closed and convex with*

$$\sup_{x\in\mathbb{R}^d}\left|\mathrm{dist}_{\mathbf{C}(t)}(x)-\mathrm{dist}_{\mathbf{C}(s)}(x)\right|\leq\kappa_{\mathbf{C}}|t-s|,$$

for some $\kappa_{\mathbf{C}}>0$*. Then there exists a unique Lipschitz solution to*

$$\begin{cases}\dot{\gamma}(t)\in-\mathcal{N}_{\mathbf{C}(t)}\left(\gamma(t)\right) & a.e.\ t\in[0,T].\\ \gamma(0)=x\in\mathbf{C}(0).\end{cases}\qquad (SP)$$

Moreover, $\|\dot{\gamma}(t)\|\leq\kappa_{\mathbf{C}}$ *for a.e.* $t\in[0,T]$*.*

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[6] J.J. Moreau: Rafle par un convexe variable I. Sém. Anal. Convexe Montpellier (1971), Exposé 15.

[7] J.J. Moreau: Rafle par un convexe variable II. Sém. Anal. Convexe Montpellier (1972), Exposé 3.

Several extensions are possible.

Let us fix $\emptyset \neq C \subset \mathbb{R}^d$.

- Assume that C is closed and let $r > 0$. We say that C is r-prox-regular if

$$(\forall\, x \in C)(\forall\, \eta \in \mathcal{N}_C(x))(\forall\, y \in C) \quad \left\langle \frac{\eta}{\|\eta\|}, y - x \right\rangle \leq \frac{1}{2r}\|x - y\|^2.$$

- One can show (see [2]) that if C is r-prox-regular then, for every $x \in C_r = \{y \in \mathbb{R}^d \,|\, d(y, C) \leq r\}$, the projection $\mathrm{proj}_C(x)$ of x onto C is unique.

[2] G. Colombo and L. Thibault : Prox-regular sets and applications. Handbook of Nonconvex Analysis and Applications (2010).

- Let $f\colon [0,T]\times\mathbb{R}^d \to \mathbb{R}^d$ and suppose that there exists $\kappa_1 \in L^1([0,T];\mathbb{R})$ such that

  $$\|f(t,x)-f(t,x')\| \le \kappa_1(t)\|x-x'\| \quad \text{for a.e. } t\in[0,T], \text{ for all } x,\, x'\in\mathbb{R}^d.$$

  Moreover, suppose that there exists a function $\kappa_2 \in L^1([0,T];\mathbb{R})$ such that

  $$\|f(t,x)\| \le \kappa_2(t)(1+\|x\|) \quad \text{for a.e. } t\in[0,T], \text{ for all } x\in\mathbb{R}^d.$$

- Let $\mathbf{C}\colon [0,T] \rightrightarrows \mathbb{R}^d$ be such that, for every $t\in[0,T]$, $\emptyset \neq \mathbf{C}(t)$ is closed and r-prox-regular. Moreover, assume the existence of $\kappa_{\mathbf{C}} > 0$ such that

  $$\sup_{x\in\mathbb{R}^d} \left|\text{dist}_{\mathbf{C}(t)}(x) - \text{dist}_{\mathbf{C}(s)}(x)\right| \le \kappa_{\mathbf{C}}|t-s|, \quad \text{for all } s,t\in[0,T].$$

Theorem (Edmond-Thibault (2005))

Under the previous assumption the perturbed sweeping process equation

$$\begin{cases} \dot{\gamma}(t) \in f(t, \gamma(t)) - \mathcal{N}_{\mathbf{C}(t)}(\gamma(t)) & a.e.\ t \in [0, T], \\ \gamma(0) = x \in \mathbf{C}(0) \end{cases} \tag{2}$$

admits a unique Lipschitz solution γ.

Remark

- *One can obtain this result as a consequence of a fixed point argument involving the non perturbed sweeping process extended to the case where $\mathbf{C}(t)$ is r-prox-regular.*
- *If, in addition, f jointly continuous, one can also approximate the solution by the following catching-up approximation:*

$$\begin{cases} \gamma_{k+1}^n = \mathrm{proj}_{\mathbf{C}(t_{k+1}^n)}\left(\gamma_k^n + \Delta t_k f(t_k, \gamma_k^n)\right), \\ \gamma_0^n = x \in \mathbf{C}(0). \end{cases}$$

Plan

Introduction

Finite horizon optimal control: the dynamic programming approach

Numerical approximation

Finite horizon optimal control

Let $T > 0$ and $x \in \mathbb{R}^d$, we deal with the following optimal control problem

$$\begin{cases} \inf \displaystyle\int_0^T \ell\,(t, \gamma(t), \alpha(t))\,\mathrm{d}t + g(\gamma(T)) \\ \text{s.t.} \quad \dot\gamma(t) \in f(t, \gamma(t), \alpha(t)) - \mathcal{N}_{\mathbf{C}(t)}(\gamma(t)) \quad \text{for a.e. } t \in (0, T), \\ \qquad \gamma(0) = x, \quad \text{with } x \in \mathbf{C}(0) \\ \qquad \alpha \in \mathcal{A}. \end{cases}$$

- Here $\mathcal{A} = \{\alpha : [0, T] \to A, \text{ measurable}\}$ denotes the set of admissible controls, with $A \subset \mathbb{R}^m$ being a nonempty compact set.
- The study of optimal control problems of the sweeping process have already been addressed, in different frameworks in [1, 4]

[1] G. Colombo and M. Palladino: The minimum time function for the controlled Moreau's sweeping process (2016).

[4] C. Hermosilla, M. Palladino, and E. Vilches: Hamilton–Jacobi–Bellman Approach for Optimal Control Problems of Sweeping Processes (2024).

Assumptions

1. $f\colon [0,T] \times \mathbb{R}^d \times \mathbb{R}^m \to \mathbb{R}^d$ is continuous and satisfies
   - There exists $\kappa_f > 0$ such that for all $t \in [0,T]$ and $a \in A$
   $$\|f(t,x,a) - f(t,y,a)\| \le \kappa_f \|x - y\| \quad \text{for all } x, y \in \mathbb{R}^d.$$
2. $\ell\colon [0,T] \times \mathbb{R}^d \times \mathbb{R}^m \to \mathbb{R}$ and $g : \mathbb{R}^d \to \mathbb{R}$ are continuous and satisfies
   - There exists $\kappa_\ell > 0$ such that for all $t \in [0,T]$ and $a \in A$
   $$|\ell(t,x,a) - \ell(t,y,a)| \le \kappa_\ell \|x - y\| \quad \text{for all } x, y \in \mathbb{R}^d.$$
   - There exists $\beta_g > 0$ and $\kappa_g > 0$ such that
   $$|g(x) - g(y)| \le \kappa_g \|x - y\| \quad \text{for all } x, y \in \mathbf{C}(T).$$

The value function

Let us define the value function $v\colon \mathrm{gr}(\mathbf{C}) \subset [0,T] \times \mathbb{R}^d \to \mathbb{R}$ by

$$v(t,x) = \inf \left\{ \int_t^T \ell\left(s,\gamma(s),\alpha(s)\right) \mathrm{d}s + g(\gamma(T)) \;\middle|\; (\alpha,\gamma) \text{ is admissible} \right\}.$$

- (Dynamic programming principle) For all $(t,x) \in \mathrm{gr}(\mathbf{C})$ and $\tau \in (t,T]$

$$v(t,x) = \inf \left\{ \int_t^\tau \ell\left(s,\gamma(s),\alpha(s)\right) \mathrm{d}s + v(\tau,\gamma(\tau)) \;\middle|\; (\alpha,\gamma) \text{ is admissible} \right\}.$$

- One can show that v is locally Lipschitz.

The HJB equation

- For every $(t,x) \in \text{gr}(\mathbf{C})$ and $p \in \mathbb{R}^d$, let us define

  $$\sigma_{\mathbf{C}}(t,x,p) = \sup\left\{\langle p,q\rangle \,\middle|\, q \in \mathcal{N}_{\mathbf{C}(t)}(x),\ \|q\| \le \kappa_C + \beta_f\big(1+\|x\|\big)\right\},$$

  where $\beta_f > 0$ is such that

  $$\|f(t,x,a)\| \le \beta_f(1+\|x\|) \quad \text{for all } (t,x) \in \text{gr}(\mathbf{C}) \text{ and } a \in A.$$

- We also set

  $$H(t,x,p) = \sup\{-\langle f(t,x,a),p\rangle - \ell(t,x,a) \,|\, a \in A\}$$

  for all $(t,x) \in \text{gr}(\mathbf{C})$ and $p \in \mathbb{R}^d$.

Let $V\colon \mathrm{gr}(\mathbf{C}) \to \mathbb{R}$ be continuous. We say that

- V is a *viscosity subsolution* to HJB if for any $(t,x) \in \mathrm{gr}(\mathbf{C})$ and $\phi \in C^1(\mathrm{gr}(\mathbf{C}))$ such that $V - \phi$ has a local maximum at (t,x) we have

$$\begin{cases} -\partial_t \phi(t,x) + H(t,x,\nabla_x \phi(t,x)) \leq 0 & \text{if } x \in \mathrm{int}(\mathbf{C}(t)), \\ -\partial_t \phi(t,x) + H(t,x,\nabla_x \phi(t,x)) - \sigma_{\mathbf{C}}(t,x,-\nabla_x \phi(t,x)) \leq 0 & \text{if } x \in \partial \mathbf{C}(t) \end{cases}$$

- V is a *viscosity supersolution* to HJB if for any $(t,x) \in \mathrm{gr}(\mathbf{C})$ and $\phi \in C^1(\mathrm{gr}(\mathbf{C}))$ such that $V - \phi$ has a local minimum at (t,x) we have

$$\begin{cases} -\partial_t \phi(t,x) + H(t,x,\nabla_x \phi(t,x)) \geq 0 & \text{if } x \in \mathrm{int}(\mathbf{C}(t)), \\ -\partial_t \phi(t,x) + H(t,x,\nabla_x \phi(t,x)) + \sigma_{\mathbf{C}}(t,x,\nabla_x \phi(t,x)) \geq 0 & \text{if } x \in \partial \mathbf{C}(t). \end{cases}$$

- V is a *viscosity solution* to HJB if it is a viscosity sub- and supersolution to HJB.

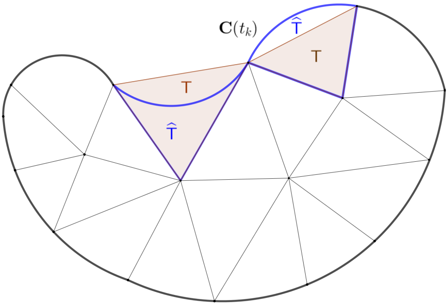
Theorem

The value function is the unique viscosity solution to the HJB equation.

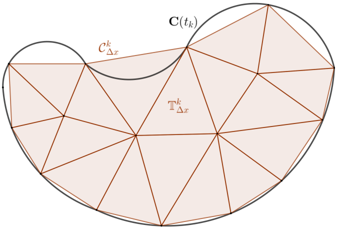
Plan

Numerical scheme

- Let $\Delta x \in]0,1[$, $\Delta t = T/N_{\Delta t}$, define $\mathcal{I}_{\Delta t} = \{0,\dots,N_{\Delta t}\}$ and set $\mathcal{I}^*_{\Delta t} = \mathcal{I}_{\Delta t} \setminus \{N_{\Delta t}\}$.
- We define a straight triangulation $\mathbb{T}^k_{\Delta x}$ composed of elements T, each of which is a straight d-simplex whose vertices coincide with those of some curved d-simplex $\widehat{\mathsf{T}} \in \widehat{\mathbb{T}}^k_{\Delta x}$.



- Let us define the polyhedral domain defined by $\mathcal{C}^k_{\Delta x} = \bigcup_{\mathsf{T} \in \mathbb{T}^k_{\Delta x}} \mathsf{T}$.



- We assume that for any $\widehat{\mathsf{T}} \in \widehat{\mathbb{T}}^k_{\Delta x}$ and $x \in \widehat{\mathsf{T}}$ we have

  $\mathrm{proj}_{\mathsf{T}}(x) \in \mathrm{argmin}_{x' \in \mathcal{C}^k_{\Delta x}} \|x - x'\|$ and we set $\mathrm{p}_{\mathcal{C}^k_{\Delta x}}(x) = \mathrm{proj}_{\mathsf{T}}(x)$.

  We will also assume that

$$\left| x - \mathrm{p}_{\mathcal{C}^k_{\Delta x}}(x) \right| \leq C_d (\Delta x)^2 \quad \text{for all } x \in \mathcal{C}^k_{\Delta x}.$$

Interpolation function

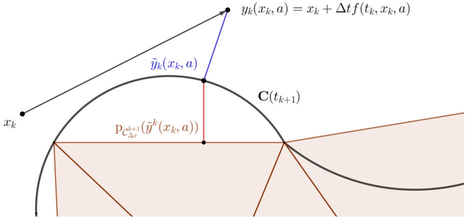
For any $k \in \mathcal{I}_{\Delta t}$ and any function $\phi : \mathcal{G}^k_{\Delta x} \to \mathbb{R}$ we define the interpolation

$$I^k[\phi](x) = \sum_{i \in \mathcal{I}^k_{\Delta x}} \beta^k_i(\mathrm{p}_{\mathcal{C}^k_{\Delta x}}(x))\phi(x^k_j) \quad \text{for all } x \in \mathbf{C}(t_k).$$

A fully discrete scheme

Given $k \in \mathcal{I}^*_{\Delta t}$, $x \in \mathbf{C}(t_k)$, $a \in A$ we define

$$y_k(x,a) = x + \Delta t f(t_k, x, a), \quad \text{and} \quad \widetilde{y}_k(x,a) = \mathrm{proj}_{\mathbf{C}(t_{k+1})}(y_k(x,a))$$



A semi-Lagrangian scheme

- For $k \in \mathcal{I}^*_{\Delta t}$ and $x_i^k \in \mathcal{G}^k_{\Delta x}$, consider the fully discrete SL scheme

$$\begin{aligned} V_k(x_i) &= \inf_{a\in A}\left\{\Delta t \ell(t_k, x_i, a) + I^{k+1}[V_{k+1}](\widetilde{y}_k(x_i, a))\right\} \\ V_{N_{\Delta t}}(x_i) &= g(x_i) \quad \text{for all } x_i \in \mathcal{G}^{N_{\Delta t}}_{\Delta x}. \end{aligned}$$

- Probabilistic interpretation of the scheme. Consider the set of policies $\Pi_{\Delta t} = \{(\pi_k)_{k\in\mathcal{I}^*_{\Delta t}} \mid \pi_k : \mathcal{G}^k_{\Delta x} \to A\}$. Given $x_i \in \mathcal{G}^k_{\Delta x}$ and $\pi \in \Pi_{\Delta t}$, we can define a Markov chain $\{X_j^{k,x_i,\pi} \mid j = k, \dots, N_{\Delta t}\}$ with transition probabilities

$$\begin{cases} \mathbb{P}(X_k^{k,x_i,\pi} = x_i) = 1 \\ \mathbb{P}(X_{m+1}^{k,x_i,\pi} = x_j^{m+1} \mid X_m^{k,x_i^k,\pi} = x_i^m) = \beta_j^{m+1}\left(\mathrm{p}_{\mathcal{C}^{m+1}_{\Delta x}}(y_m(x_i^m, \pi_m(x_i^m)))\right). \end{cases}$$

- Then for every $k \in \mathcal{I}^*_{\Delta t}$ and $x_i \in \mathbf{C}(t_k)$ we have

$$V_k(x_i) = \inf_{\pi \in \Pi_{\Delta t}} \mathbb{E} \left(\Delta t \sum_{j=k}^{N_{\Delta t}-1} \ell(t_j, X_j^{k,x_i,\pi}, \pi_j(X_j^{k,x_i,\pi})) + g(X_{N_{\Delta t}}^{k,x_i,\pi}) \right).$$

Given the sequences $(\Delta t_n)_n$ and $(\Delta x_n)_n$ we set $\Delta_n = (\Delta t_n, \Delta x_n)$.

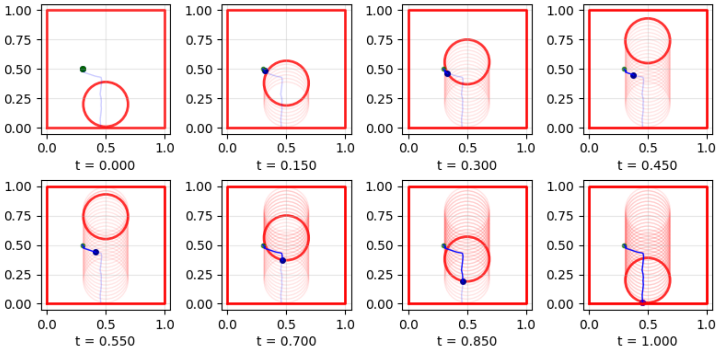
Theorem (Uniform convergence over compact sets)

Let $K \subseteq gr(\mathbf{C})$ be compact. Assume that $\Delta_n \to (0,0)$ when $n \to \infty$ with $(\Delta x_n)^2/\Delta t_n \to 0$. Then

$$\sup \left\{ \left| V_k(x_i^k) - v(t_k, x_i^k) \right| \;\middle|\; k \in \mathcal{I}_{\Delta t_n},\ x_i^k \in \mathcal{G}^k_{\Delta x_n},\ (t_k, x_i^k) \in K \right\} \underset{n\to\infty}{\longrightarrow} 0.$$

A numerical example

- We take $T = 1$, $N = 40$, $\Delta x = 0.05$, $\ell(t, x, a) = \|a\|^2/2$, $g(x) = \|x - (0, 0.5)\|^2/2$, and $f(t, x, a) = a$.



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